:

Consensus Analysis and Synthesis of Networked Multi-Agent Systems

Weisong Zhong

Ph.D. Thesis

June 2012

Consensus Analysis and Synthesis of Networked Multi-Agent Systems

by Weisong Zhong

Prifysgol Morgannwg

The University of Glamorgan

A thesis submitted in partial fulfilment of the

requirement for the degree of Doctor of Philosophy

June 2012

Abstract

As one of the major fields, Networked Multi-Agent Systems (NMAS), which deals with the study of how network architecture and interactions among network components influence global control goals, has received widely attention across science and engineering. A usual problem that appears in the coordination of NMAS is the consensus problem, i.e., given initial values (scalar or vector) of agents, establish conditions under which through local interactions and computations, the agents asymptotically achieve some kind of agreement, such as control of mobile vehicles, information processing in sensor networks and design of distributed algorithms. Researches mainly focused on the analysis of NMAS consensus in the past, the interconnection topology and some consensus algorithms were given in advance and the relevant research objective was to verify whether the states of all agents converge to some common value. Agents with single integrator dynamics, double integrator dynamics or more complicated dynamics are the present research emphasis. In addition, most of present research activities focus on theoretical study of consensus problems based on relatively simple simulation experiments, but it will still be a key element of research in the future.

Although it is more complicated to consider consensus problems for a team of agents with more complex nonlinear dynamics and even heterogenous dynamics, such kind of problems are very important. To the best of our knowledge, there are few effective results on this topic. Usually, no common equilibrium for all agents exists even if each isolated agent has an equilibrium, but an NMAS with non-identical agents may still exhibit some consensus behaviors which are from being fully understood. Certain reasonable and satisfactory boundedness of state motion errors between different agents can be taken as useful consensus properties.

This thesis focuses on the global consensus problems of NMAS consisting of identical or non-identical agent dynamics, and the proposed consensus property is formulated in terms of certain boundedness of state errors. Moreover, on special occasions, we still investigate their exact consensus conditions. Compared with many existing results, the thesis makes several significant contributions. Firstly, we generalize the related results for the case of identical agent dynamics to the case of non-identical agent dynamics and the proposed results cover the existing criteria of networks with identical agent dynamics as special cases. Secondly, we consider the communication delay among the agents, global consensus criteria are given based on solving a number of lower dimensional matrix inequalities and scalar inequalities, which generalize the criteria using the method of the master stability function for NMAS with identical agents. Finally, globally bounded consensus conditions for both delay-independent and delay-dependent conditions based on the Lyapunov-Krasovskii functional method are derived.

Acknowledgements

I would like to express my sincere gratitude to my supervisor, Professor Guoping Liu, for his patient guidance and invigorative encouragement. He has created harmonious working environment and good atmosphere of academic research here, and he actually practises what he preaches, and these scenarios have helped me reach my full potential. The regular discussions and his invaluable comments on my work have enabled me to be more familiar with the NMAS area. I also want to thank my second supervisor, Dr. David Rees, for his patient and serious-minded English correcting in my publications. My warm thanks are due to my third Supervisor Mr. Clive Thomas, for his patient English teaching and broad engineering knowledge.

I greatly appreciate the help of my colleagues Prof. Jun Zhao, Prof. Ximing Sun, Dr. Tao Liu, Dr. Yunbo Zhao, Dr. Wenshan Hu, Dr. Jian Sun, Dr. Xiaopeng Liu, Ms. Rongni Yang, Dr. Peng Li, Dr. Hongbo Song, Mr. Xiao Guo, Dr. Shuang Li, Ms. Ruijuan Liu, Ms. Nurul Adilla Mohd Subha, Dr. Jun Liang, Dr. Bo Shen, Dr. Hongyi Li, etc, who helped me when I'm studying in the University of Glamorgan. Their valuable discussion and kindly help are greatly appreciated. Thanks also go to the Faculty of Advanced Technology, University of Glamorgan who provides me with the Research studentship award and financial support.

I would like to take this opportunity to thank all my friends in Trefforest who have given me support and encouragement.

Finally, my deepest and foremost gratitude is dedicated to my family for their constant support and understanding, without their emotional supports and helpful assistance, this work could not have been finished.

Table of Contents

Chapter 1

Introduction

1.1 Networked Multi-Agent Systems

Networked Multi-Agent Systems (NMAS) are normally consisted of a large quantity of simple systems interacting through communication channels and many systems widely found in the fields of sciences and engineering can be modeled accordingly, such as satellite communications, GPS, robot networks, biological networks, sensor networks, unmanned vehicles, power systems, animal cooperative aggregation, schools of fish, and flocks of birds, etc [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] .

Each agent in an NMAS has its own distributed knowledge, capabilities or skills when performing specific actions. However, it is unusual and even useless for an isolated agent to act individually despite of the common loosely-coupled network topology. All agents in an NMAS are expected to be situated in the same environment and they can communicate through a series of interaction protocols. And therefore, NMAS can be used to model many existing complex systems and its corresponding research can bring us new methods to deal with problems which can't be resolved by any one individual agent.

As for the advantages related to the usage of the NMAS technology, there are so many good properties compared with many other available methods, such as reliability, flexibility, robustness, extensibility, maintainability and reusability, etc. At the same time, the NMAS technology can reduce the expenses for establishment, operation and maintenance of the system tremendously based on its computational efficiency and speed.

However, the research on NMAS is also confronted with many difficulties especially for its design and implementation. How to model and identify each isolated agent's exact dynamics and its corresponding NMAS's inner and outer exact coupling topology structure; How to formulate or decompose the relevant tasks and objectives; How to represent the information about environment, actions and knowledge; How to design efficient and effective protocols, planning and learning algorithms according to some specific performance indices, etc.

1.2 Research Review

NMAS analysis involves the study of how network architectures and interactions among network components influence global objectives. NAMS synthesis involves the generation of a desired collective behavior by local interaction protocols among the agents. The main research on NMAS can be categorized into two areas: One is to design distributed estimation techniques for the sensor networks, and the other is to control mobile autonomous agents by using information obtained over the network [23]. In both areas some important contributions have been made in recent years [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

The consensus problem requires an agreement to be achieved that depends on the states of all agents, i.e., to design distributed control strategies based on local information that enables all agents to reach some kind of agreements on certain quantities of interest. The topic has been studied across many fields of science and engineering and many results have been obtained. The consensus ability of NMAS is usually involved with each agent's isolated dynamics and its connection topology structure. Once the isolated agent dynamics are determinated, the consensus ability of NMAS depends on its connection topology structure. For a NMAS, its connection topology is usually described by the graph theory. Based on the graph theory and the special characteristics of NMAS, such system can be usually decoupled into a series of lower-dimensional systems by means of suitable coordinates transformations, thus the consensus problems are solvable if the stability of the lower-dimensional systems can be guaranteed. In the context of NMAS, many pioneering contributions involved with various distributed strategies that achieve consensus have been witnessed. Olfati-Sabre introduced two consensus criteria for networks with and without time-delays and provides convergence analysis for three kinds of MAS with fixed and switching topologies [35]. A passivity-based design framework developed to process the group coordination problem, with both fixed and time-varying communication structures, has also been considered [36]. All agents reach a consensus if a small fraction of them are controlled by simple feedback control is demonstrated in [37]. The robust consensus problems of second-order NMAS with diverse input delays are investigated and decentralized consensus conditions are obtained for the NMAS with symmetric coupling weights based on frequency-domain analysis in [38]. The consensus problem for directed NMAS with external disturbances and model uncertainties for fixed and switching topologies is discussed in [39]. The average consensus problem for undirected NMAS having communication delays is studied and sufficient conditions are provided for the existence of average consensus under bounded communication delays in [40]. A distributed algorithm that asymptotically achieved consensus is characterized and two discontinuous distributed algorithms that achieve maximum and minimum consensus are provided respectively in [41]. The results stated above are often based on suitable coordinates transformations, and these transformations decoupled the original system into several lower-dimensional dynamic systems. Having realized that the communication topology plays a key role in forming its collective behaviors, a verity of connection topology have been investigated to better understand how the topology structure influences consensus behavior and such results include the time-varying connection topology, switching connection topology(slow-varying and fast-varying), time delay in communication channels, nonlinear coupling, stochastic coupling, uncertain coupling, etc [42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]. It should be noted that the agent dynamics involved in most existing results are often restricted to be linear and identical ones. However, this is not always the case in practice and significant differences exist widely within the relevant agents. Strictly speaking, each agent, regardless of the similarity in its main functions, have characteristics that exhibit a degree of difference, especially in their isolated agent dynamics which are usually modeled as nonlinear dynamical systems.

1.3 Motivations and Objectives

The consensus analysis of a NMAS consisting of identical or non-identical nonlinear dynamics is much more complicated than the identical or linear case and few results have been reported up to now. However, the following idea widely used in the Complex Dynamical Network (CDN) can be applied to deal with the consensus analysis of the NMAS. The similarity between the consensus of NMAS and the synchronization of CDN suggests a way forward [27, 53]. A CDN is a large set of interconnected dynamic nodes where its specific representation is determined by the specific application. It has attracted tremendous attention in recent years [54, 55]. Since the connection topology plays a key role in forming the behaviors of a CDN, researchers have examined a variety of connection topologies and tried to better understand how topology influences the network behavior. Synchronization is one of the key issues that affect network behavior and has been extensively addressed and a large number of papers on this topic have appeared based on CDN with identical nodes [56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85]. As for the synchronization of CDN with non-identical nodes, some results have been proposed. A simulation based synchronization study for non-identical Kuramoto oscillators was carried out in [86]. A simple case where all non-identical nodes have the same equilibrium was considered in [87] and a synchronization criterion was given by constructing the same Lyapunov function for all the nodes. [88, 89] studied the synchronization problem for a CDN with non-identical nodes and the proposed results extend the relevant asymptotic synchronization criteria to this case. Several collective properties for coupled non-identical chaotic systems were discussed respectively in [90, 91]. Therefore, if the ideas in the synchronization problem of CDN are applied properly, then consensus problems are solvable.

Inspired by these early results, the present research will focus on the global consensus problems of NMAS, and the proposed consensus property is formulated in terms of certain boundedness of state errors. The behavior of the NMAS with non-identical agent dynamics is much more complicated than the identical case. Usually, no common equilibrium for all agents exists even if each agent has an equilibrium, neither does a consensus manifold exist in the classical sense. Consensus of a NMAS with identical agents is usually described in terms of (asymptotically) identical dynamical evolution of state variables of every agent in the

NMAS, which is easy to understand. However, this collective behavior, called exact consensus no longer exists in the NMAS with non-identical agents due to the difference between the dynamics of the agents. Furthermore, we can't decompose the NMAS with non-identical agent dynamics into a number of lower dimensional systems exactly like the identical-agent case. Yet, a NMAS with non-identical agents may still exhibit some kinds of consensus behaviors which are far from being fully understood, and very few results have been reported by now. Certain reasonable and satisfactory boundedness of state motions errors between different agents can be taken as useful consensus properties.

1.4 Outline of Concrete Research Content

The remaining parts of the thesis are outlined as follows.

Chapter 2 Controlled Global Exact Consensus of Multi-Agent Systems with Communication Time Delay

In this chapter, the global exact consensus problem of NMAS consisting of nonlinear, identical agent dynamics and communication time-delay topology are investigated. Based on linear feedback and adaptive feedback, global exact consensus criteria has been obtained respectively. Furthermore, the relevant pinning control criteria is also derived and the proposed results are theoretically proved to be effective based on the Lyapunov-Krasovskii functional method.

The presentation of this chapter is mainly based on:

[3] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus of Multi-Agent Systems with Communication Time Delay. *In the 2nd International Conference on Intelligent Control and Information Processing*, pages 775-778, Harbin, China, 25-28 July, 2011.

[13] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Global Exact Consensus

of Multi-Agent Systems with Communication Time Delay. *International Journal of Robust and Nonlinear Control*. Submitted.

Chapter 3 Controlled Consensus Criteria for NMAS with Uncertain Coupling

In this chapter, the consensus problem for uncertain NMAS will be discussed. Based on the network connection topology and Lyapunov stability theory, the local and global decentralized consensus for such systems will be investigated respectively. Under the different assumptions, several consensus criteria are obtained.

The presentation of this chapter is mainly based on:

[4] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus Criteria for a Class of Uncertain Multi-Agent Systems. *In the 18th IFAC World Congress*, pages 5448-5452, Milano, Italy, 28 August- 2 September, 2011.

[15] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus Criteria for NMAS with Uncertain Coupling. *IET Control Theory & Applications*. Submitted.

Chapter 4 Controlled Consensus of NMAS with Non-Identical Agent Dynamics

The global consensus problem of NMAS with different agent dynamics will be investigated in this chapter. The proposed consensus property is formulated in terms of certain boundedness of state errors and the exact consensus is achieved by using nonlinear controllers. Based on Lyapunov stability theorem, the bounded and exact consensus criterion has been proved systematically.

The presentation of this chapter is mainly based on:

[5] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus of Multi-Agent Systems with Different Agent Dynamics. *In the UKACC International Conference on Control 2010*, pages 542-547, Coventry, UK, 7-10 September, 2010.

[15] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus of Multi-Agent Systems with Non-Identical Agent Dynamics. *IET Control Theory & Applications*. Submitted.

Chapter 5 Global Bounded Consensus of NMAS with Non-Identical Agents and Communication Time-Delay Topology

In this chapter, the global bounded consensus problem of NMAS exhibiting nonlinear, non-identical agent dynamics and communication time-delay will be investigated. Globally bounded consensus conditions for both delay-independent and delay-dependent conditions based on the Lypunov-Krasovskii functional method are derived. In addition, globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived.

The presentation of this chapter is mainly based on:

[6] W.S. Zhong, G.P. Liu, D. Rees and C. Thomas. Global Bounded Controlled Consensus of Networked Multi-Agent Systems with Non-Identical Dynamical Agents. *In the 16th International Conference on Automation & Computing*, pages 186-191, Birmingham, UK, 11 September, 2010.

[7] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Controlled Consensus of Networked Multi-Agents Systems with Non-Identical Dynamical Agents. *In the 18th IFAC World Congress*, pages 245-250, Milano, Italy, 28 August- 2 September, 2011.

[2] W.S. Zhong, G.P. Liu and D. Rees. Global Bounded Consensus of Multi-Agent Systems with Non-Identical Nodes and Communication Time-Delay Topology. *International Journal of Systems Science*, DOI:10.1080/00207721.2011.601346.

[9] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Controlled Consensus of Multi-Agents Systems with Non-Identical Nodes and Communication Time-Delay Topology. *In the UKACC International Conference on Control 2012*, Cardiff, UK, 3-5 September, 2012. Accepted

[12] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Controlled Consensus of Delayed Multi-Agents Systems with Non-Identical Nodes. *IEEE Transactions on Automatic Control*. Submitted.

Chapter 6 Global Consensus Analysis of NMAS with Different Agent Dynamics and Time-Varying Delay Topology

In this chapter, the global bounded consensus problem of NMAS exhibiting nonlinear, non-identical agent dynamics with communication time-varying delay will be investigated. Delay-independent and delay-dependent bounded consensus criterion and controlled bounded consensus criterion based on the Lypunov-Krasovskii functional method and pinning control scheme are derived.

The presentation of this chapter is mainly based on:

[3] W.S. Zhong, G.P. Liu and C. Thomas. Global Consensus Analysis of Networked Multi-Agent Systems with Heterogeneous Dynamics and Time-Varying Communication Delay. *In the 8th Japan-China International Workshop on Internet Technology and Control Applications*. Tokyo, Japan, 5-11 December, 2011.

[8] W.S. Zhong, G.P. Liu and C. Thomas. Global Consensus Analysis of Networked Multi-Agent Systems with Heterogeneous Dynamics and Time-Varying Communication Delay. *In the 8th Japan-China International Workshop on Internet Technology and Control Applications*. Tokyo, Japan, 5-11 December, 2011.

[10] W.S. Zhong, G.P. Liu and C. Thomas. Global Controlled Consensus of Multi-Agent Systems with Different Agent Dynamics and Time-Varying Communication Delay. *In the UKACC International Conference on Control 2012*, Cardiff, UK, 3-5 September, 2012. Accepted

[11] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Controlled Consensus of Multi-Agents Systems with Non-Identical Nodes and Communication Time-Delay Topology. *IEEE Transactions on Circuits and Systems-II*. Submitted.

Chapter 7 Global Bounded Consensus of NMAS with Different Agents

and Time Delays In this chapter, the global bounded consensus problem of NMAS consisting of nonlinear, non-identical node dynamics with communication delays will be investigated. The globally bounded controlled consensus conditions for both delay-independent and delay-dependent conditions based on the Lypunov-Krasovskii functional method are obtained.

The presentation of this chapter is mainly based on:

[1] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Consensus of Multi-Agent Systems with Non-Identical Nodes and Time Delays. *IEEE Transactions on Systems, Man, and Cybernetics-Part B*, to appear.

Chapter 8 Conclusions

The conclusions and future work will be included in this chapter.

Appendix Published and Finished Papers

In this part, all papers relevant to the author's doctoral research will be listed. The published and accepted papers are also attached as an appendix.

Part A

State Consensus Criteria of NMAS with Identical Agent Dynamics and Communication Time Delay

Chapter 2

Controlled Global Exact Consensus of Multi-Agent Systems with Communication Time Delay

This chapter investigates the global exact consensus problem of NMAS consisting of nonlinear, identical agent dynamics and communication time-delay topology. Based on linear feedback and adaptive feedback, global exact consensus criteria has been obtained respectively. Furthermore, the relevant pinning control criteria is also derived and the proposed results are theoretically proved to be effective based on the Lyapunov-Krasovskii functional method. The obtained consensus criteria ensure that all agents eventually move along the desired dynamic trajectory which is derived from the average dynamics of all agents here, meanwhile, the consensus criteria gotten here generalize many existing results which can be viewed as an extension of such relevant results. The effectiveness of the results is also demonstrated through a numerical simulation finally.

2.1 Introduction

Most research in consensus problems usually assume that the final consensus value to be a constant, which may not be the case in the sense that the information state of each agent may be dynamically evolving in time according to some inherent dynamics. It is interesting to study controlled consensus problems where the final consensus value evolves with time or as a function of environmental dynamics. This chapter will focus on the global exact consensus problems of NMAS with nonlinear agent dynamics and communication delay. Compared with many existing results, this chapter makes two significant advances. One is that we choose the average agent dynamics as the desired moving trajectories instead of a constant, and the other is we introduce linear feedback control, adaptive control and pinning control to guarantee exact consensus of the NMAS respectively.

This chapter is organized as follows. A controlled continuous-time NMAS model with nonlinear agent dynamics and communication time-delay is presented and some mathematical preliminaries are introduced in Section 2.2. The main results including linear feedback control and adaptive feedback control exact consensus criteria are derived in Section 2.3 and 2.4 respectively. In Section 2.5, a numerical simulation example is given to verify the effectiveness of the proposed results, followed by conclusions in Section 2.6.

2.2 Model Description

Consider a NMAS consisting of N agents with communication delay:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j \in \mathbb{N}_i} c_{ij} \Gamma x_j(t-\tau) + u_i, i = 1, 2, \cdots, N, \tag{2.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f(x_i(t))$ is a continuously differentiable vector function, $u_i \in R^n$ is a local controller to be designed, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix. The adjacency matrix $C = (c_{ij}) \in R^{N\times N}$, representing the communication topology relation of the NMAS, is symmetric and irreducible, $c_{ij} \geq 0$ and $c_{ii} = -\sum_{j\neq i} c_{ij}$. $\tau > 0$ is a constant time delay.

The desired moving trajectory is chosen as

$$\bar{x}(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t). \tag{2.2}$$

The consensus problem is solvable if the states of all agents satisfy $x_i(t) - x_j(t) \to 0$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$.

Define the error vector $e_i(t) = x_i(t) - \bar{x}(t)$, together with the NMAS (2.1) results in the error system in terms of $e_i(t)$:

$$\dot{e}_i(t) = f(x_i(t)) - f(\bar{x}(t)) + \sum_{j\in\mathbb{N}_i} c_{ij}\Gamma e_j(t-\tau) + f(\bar{x}(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t)) + u_i \tag{2.3}$$

The exact consensus problem of the NMAS (2.1) and the stabilization problem of the error system (2.3) are equivalent to each other, so the stability of system (2.3) is discussed, which will solve the consensus problem indirectly.

Assumption 2.1 Suppose there exists a positive constant L such that

$$\|f(x_i(t)) - f(\bar{x}(t))\| \leq L\|e_i(t)\| \tag{2.4}$$

holds for $i = 1, 2, \cdots, N$.

2.3 Global Consensus via Liner Feedback Control

Global exact consensus problem will be investigated in this section by applying local feedback control and pinning control.

Theorem 2.1. Suppose Assumption 2.1 holds and then the NMAS (2.1) achieves global exact consensus under the set of controllers

$$u_i = K_i(x_i(t) - \bar{x}(t)), \quad i = 1, 2, \cdots, N, \tag{2.5}$$

where $K_i + L + \lambda - c_{ii}(\|\Gamma^2\| + 1) \leq 0$ and $\lambda > 0$.

Proof: Choose the Lyapunov functional candidate as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N} c_{ii}\int_{t-\tau}^{t} e_i^T(\theta)\Gamma^2 e_i(\theta)d\theta, \tag{2.6}$$

then the time derivative of $V(t)$ along the solution of the error system (2.3) is given as follows

$$\begin{aligned}\dot{V}(t) =& \sum_{i=1}^{N} e_i^T(t)[f(x_i(t)) - f(\bar{x}(t))] + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{ij}e_i^T(t)\Gamma e_j(t-\tau) \\ &+ \sum_{i=1}^{N} e_i^T(t)(f(\bar{x}(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t))) + \sum_{i=1}^{N} e_i^T(t)u_i - \sum_{i=1}^{N} c_{ii}e_i^T(t)\Gamma^2 e_i(t) \\ &+ \sum_{i=1}^{N} c_{ii}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau).\end{aligned}$$

Since $\sum_{i=1}^{N} e_i^T(t)(f(\bar{x}(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t))) = 0$, together with the condition (2.4) and the controllers (2.5), then we have

$$
\begin{aligned}
\dot{V}(t) \leq & \sum_{i=1}^{N}(-\lambda + c_{ii})e_i^T(t)e_i(t) + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{ij}e_i^T(t)\Gamma e_j(t-\tau) \\
& + \sum_{i=1}^{N} c_{ii}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau) \\
& \leq -\lambda \sum_{i=1}^{N} e_i^T(t)e_i(t) + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ji}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau) \\
& + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{ij}e_i^T(t)\Gamma e_j(t-\tau) - \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ij}e_i^T(t)e_i(t)
\end{aligned}
$$

$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ji}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau) = \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ij}e_j^T(t-\tau)\Gamma^2 e_j(t-\tau)$ is satisfied according to the symmetry of adjacency matrix C, then we have the differential coefficient of V as

$$
\begin{aligned}
\dot{V}(t) \leq & -\lambda \sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} |c_{ij}| \| e_i(t) - \frac{\Gamma}{2}e_j(t-\tau)\|^2 \\
& - \frac{3}{4}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} |c_{ij}| e_i^T(t-\tau)\Gamma^2 e_i(t-\tau),
\end{aligned}
$$

thus we have

$$
\dot{V}(t) \leq -\lambda \sum_{i=1}^{N} e_i^T(t)e_i(t),
$$

in consequence, the NMAS (2.1) achieves global exact consensus under the controllers (2.5). This completes the proof.

Now, we apply the feedback control strategy on a small fraction δ $(0 < \delta \leq 1)$ of the agents in system (2.1). Suppose that nodes $i_1, i_2, \cdots, i_l$ are selected to be under control, where $l = [\delta N]$ stands for the smaller but nearest integer to the real number δN. This controlled NMAS can be described as

$$\begin{cases} \dot{x}_{i_k} = f(x_{i_k}) + \sum_{j\in\mathcal{N}_i} c_{i_kj}\Gamma x_j(t-\tau) + u_{i_k}, 1 \le k \le l, \\ \dot{x}_{i_k} = f(x_{i_k}) + \sum_{j\in\mathcal{N}_i} c_{i_kj}\Gamma x_j(t-\tau), l+1 \le k \le N. \end{cases} \tag{2.7}$$

Without loss of generality, rearrange the order of the agents in the NMAS, and let the first l be controlled, then the pinning controlled NMAS can be described by

$$\begin{cases} \dot{x}_i = f(x_i) + \sum_{j\in\mathcal{N}_i} c_{ij}\Gamma x_j(t-\tau) + u_i, 1 \le i \le l, \\ \dot{x}_i = f(x_i) + \sum_{j\in\mathcal{N}_i} c_{ij}\Gamma x_j(t-\tau), l+1 \le i \le N. \end{cases} \tag{2.8}$$

Corollary 2.1 Suppose Assumption 2.1 holds then the NMAS (2.1) achieves global exact consensus under the pinning controllers

$$\begin{cases} u_i = -K_i(x_i(t) - \bar{x}(t)), & 1 \le i \le l, \\ u_i = 0, & l+1 \le i \le N, \end{cases} \tag{2.9}$$

where $-K_i + L + \lambda - c_{ii}(\|\Gamma^2\| + 1) \le 0$ and $\lambda > 0$.

Proof: Choose the same Lyapunov functional candidate as (2.6), then derive $V(t)$ along the trajectories of the closed-loop NMAS obtained by combining (2.8) and (2.9), the remainder proof is similar to that of the Theorem 2.1, so is omitted here. This completes the proof.

Remark 2.1 In fact, K_i can be chosen as matrixes in the Theorem 2.1 and the Corollary 2.1, and the condition of the Theorem 1 and the Corollary 2.1 can be further weakened as $K_i + LI_n + \lambda I_n - c_{ii}(\Gamma^2 + I_n) \le 0$ and $-K_i + LI_n + \lambda I_n - c_{ii}(\Gamma^2 + I_n) \le 0$ respectively, where I_n is a unitary matrix of n-dimensional.

2.4 Global Exact Consensus via Adaptive Feedback Control

In this section, controlled global exact consensus problem will be discussed based on direct adaptive control and adaptive pinning control.

Theorem 2.2. Suppose Assumption 2.1 holds and then the NMAS (2.1) achieves global exact consensus under the set of adaptive controllers

$$\begin{cases} u_i = k_i(t)(x_i(t) - \bar{x}(t)), \\ \dot{k}_i(t) = h_i e_i^T(t) e_i(t), \ \ h_i > 0, \quad i = 1, 2, \cdots, N, \end{cases} \tag{2.10}$$

where $L - h_i + c_{ii}(\|\Gamma^2\| + 1) + \lambda \leq 0$ and $\lambda > 0$.

Proof: Choose the Lyapunov functional candidate as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N} c_{ii} \int_{t-\tau}^{t} e_i^T(\theta)\Gamma^2 e_i(\theta) d\theta + \frac{1}{2}\sum_{i=1}^{N} \frac{(k_i(t) - h_i)^2}{h_i}.$$

The derivative of V with respect to time t along the trajectories of (2.3) is then given by

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^{N} e_i^T(t)[f(x_i(t)) - f(\bar{x}(t))] + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{ij} e_i^T(t)\Gamma e_j(t-\tau) \\ & + \sum_{i=1}^{N} e_i^T(t)(f(\bar{x}(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t))) + \sum_{i=1}^{N} e_i^T(t)u_i - \sum_{i=1}^{N} c_{ii} e_i^T(t)\Gamma^2 e_i(t) \\ & + \sum_{i=1}^{N} c_{ii} e_i^T(t-\tau)\Gamma^2 e_i(t-\tau) + \sum_{i=1}^{N} \frac{k_i(t) - h_i}{h_i}\dot{k}_i(t). \end{aligned}$$

The remainder proof is very similar to that of the Theorem 2.1, so is omitted here.

At last, we still have

$$\dot{V}(t) \leq -\lambda \sum_{i=1}^{N} e_i^T(t)e_i(t),$$

which means that the NMAS (2.1) achieves global exact consensus under the controllers (2.10). This completes the proof.

Similar to the analysis of the Corollary 2.1, we introduce pinning controllers to some agents, then we have the following corollary:

Corollary 2.2 Suppose Assumption 2.1 holds and then the NMAS (2.1) achieves global exact consensus under the set of controllers

$$\begin{cases} u_i = k_i(t)(x_i(t) - \bar{x}(t)), \dot{k}_i(t) = h_i e_i^T(t)e_i(t), \ h_i > 0, & 1 \leq i \leq l, \\ u_i = 0, & l+1 \leq i \leq N \end{cases} \tag{2.11}$$

where $L - h_i + c_{ii}(\|\Gamma^2\| + 1) + \lambda \leq 0$ and $\lambda > 0$.

Proof: Choose the Lyapunov functional candidate as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N} c_{ii} \int_{t-\tau}^{t} e_i^T(\theta)\Gamma^2 e_i(\theta)d\theta + \frac{1}{2}\sum_{i=1}^{l} \frac{(k_i(t) - h_i)^2}{h_i}.$$

The remainder proof is similar to that of the Theorem 2.1, so is omitted here. This completes the proof.

The above result can be generalized to the case of global exponential consensus, and the corresponding result is given as follows.

Theorem 2.3 Suppose Assumption 2.1 holds and then the NMAS (2.1) achieves global exponentially consensus under the set of controllers

$$\begin{cases} u_i = k_i(t)(x_i(t) - \bar{x}(t)), \dot{k}_i(t) = h_i e_i^T(t) e_i(t) \mathtt{exp}(\mu t), & 1 \le i \le l, \\ u_i = 0, & l+1 \le i \le N \end{cases} \tag{2.12}$$

where $L + \frac{\mu}{2} - c_{ii}(\|\Gamma^2\| \mathtt{exp}(\mu\tau) + 1) - h_i + \lambda \le 0$ and $\lambda > 0$.

Proof: Choose the Lyapunov functional candidate as

$$\begin{aligned} V(t) =& \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) e_i(t) \mathtt{exp}(\mu t) \\ & - \sum_{i=1}^{N} c_{ii} \int_{t-\tau}^{t} e_i^T(\theta) \Gamma^2 e_i(\theta) \mathtt{exp}(\mu(\theta + \tau)) d\theta + \frac{1}{2} \sum_{i=1}^{l} \frac{(k_i(t) - h_i)^2}{h_i}, \end{aligned} \tag{2.13}$$

then the time derivative of $V(t)$ along the solution of the error system (2.3) is given as follows

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^{N} e_i^T(t)[f(x_i(t)) - f(\bar{x}(t))] \mathtt{exp}(\mu t) + \sum_{i=1}^{N} \sum_{j \in \mathbb{N}_i} c_{ij} e_i^T(t) \Gamma e_j(t - \tau) \mathtt{exp}(\mu t) \\ & + \sum_{i=1}^{N} e_i^T(t)(f(\bar{x}(t)) - \frac{1}{N} \sum_{i=1}^{N} f(x_i(t))) \mathtt{exp}(\mu t) - \sum_{i=1}^{N} d_i e_i^T(t) e_i^T(t) e_i(t) \mathtt{exp}(\mu t) \\ & + \frac{\mu}{2} e_i^T(t) e_i(t) \mathtt{exp}(\mu t) - \sum_{i=1}^{N} c_{ii} e_i^T(t) \Gamma^2 e_i(t) \mathtt{exp}(\mu\tau) \mathtt{exp}(\mu t) \\ & + \sum_{i=1}^{N} c_{ii} e_i^T(t - \tau) \Gamma^2 e_i(t - \tau) \mathtt{exp}(\mu t) + \sum_{i=1}^{l} (d_i - h_i) e_i^T(t) e_i(t) \mathtt{exp}(\mu t). \end{aligned}$$

Since $\sum_{i=1}^{N} e_i^T(t)(f(\bar{x}(t)) - \frac{1}{N} \sum_{i=1}^{N} f(x_i(t))) \mathtt{exp}(\mu t) = 0$, together with the condition (2.4) and the controllers (2.12), then we have

$$\begin{aligned}
\dot{V}(t) \leq & \sum_{i=1}^{N}(-\lambda + c_{ii})e_i^T(t)e_i(t)\mathtt{exp}(\mu t) \\
& + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{ij}e_i^T(t)\Gamma e_j(t-\tau)\mathtt{exp}(\mu t) + \sum_{i=1}^{N} c_{ii}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau)\mathtt{exp}(\mu t) \\
& \leq -\lambda\sum_{i=1}^{N} e_i^T(t)e_i(t)\mathtt{exp}(\mu t) + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ji}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau)\mathtt{exp}(\mu t) \\
& + \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} c_{ij}e_i^T(t)\Gamma e_j(t-\tau)\mathtt{exp}(\mu t) - \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ij}e_i^T(t)e_i(t)\mathtt{exp}(\mu t)
\end{aligned}$$

$\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ji}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau)\mathtt{exp}(\mu t) = \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ij}e_j^T(t-\tau)\Gamma^2 e_j(t-\tau)\mathtt{exp}(\mu t)$ is satisfied according to the symmetry of adjacency matrix C, then we have the differential coefficient of V as

$$\begin{aligned}
\dot{V}(t) \leq & -\lambda\sum_{i=1}^{N} e_i^T(t)e_i(t)\mathtt{exp}(\mu t) - \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}|c_{ij}|\|e_i(t) - \frac{\Gamma}{2}e_j(t-\tau)\|^2\mathtt{exp}(\mu t) \\
& - \frac{3}{4}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}|c_{ij}|e_i^T(t-\tau)\Gamma^2 e_i(t-\tau)\mathtt{exp}(\mu t),
\end{aligned}$$

thus we have

$$\dot{V}(t) \leq -\lambda\sum_{i=1}^{N} e_i^T(t)e_i(t)\mathtt{exp}(\mu t),$$

it follows that $V(t) < V(0)$ for $t \geq 0$.

According to Lyapunov function (2.13), we have $\frac{1}{2}e_i^T(t)e_i(t)\mathtt{exp}(\mu t) \leq V(t) < V(0)$. Then, we get $e_i^T(t)e_i(t) < \sqrt{2V(0)}\mathtt{exp}(-\frac{\mu}{2}t)$. Thus the NMAS (2.1) achieves global exact consensus under the controllers (2.12). This completes the proof.

2.5 An Example

Consider a NMAS constructed with 11 Lorenz chaotic systems and each agent dynamic is given by

$$\begin{cases} \dot{x}_1(t) = -10x_1(t) + 10x_2(t) \\ \dot{x}_2(t) = 28x_1(t) - x_2(t) - x_1(t)x_3(t) \\ \dot{x}_3(t) = -\frac{8}{3}x_3(t) + x_1(t)x_2(t) \end{cases}$$

The coupling matrix $C = (C_1^T C_2^T \cdots C_{11}^T)$, $C_1 = (-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1)$, $C_2 = (1\ -8\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1)$, $C_3 = (1\ 1\ -6\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1)$, $C_4 = (0\ 1\ 1\ -5\ 0\ 1\ 1\ 0\ 0\ 1\ 0)$, $C_5 = (1\ 1\ 0\ 0\ -6\ 0\ 1\ 1\ 1\ 1\ 0)$, $C_6 = (1\ 0\ 0\ 1\ 0\ -5\ 1\ 0\ 1\ 1\ 0)$, $C_7 = (1\ 1\ 0\ 1\ 1\ 1\ -7\ 1\ 0\ 1\ 0)$, $C_8 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ -5\ 0\ 1\ 1)$, $C_9 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ -6\ 1\ 1)$, $C_{10} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ -10\ 1)$, $C_{11} = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ -6)$. $\Gamma = diag\{3, 3, 3\}$, $\bar{x}(t) = \frac{1}{11}\sum_{i=1}^{N} x_i(t)$, $\lambda = 1$, $\tau = 0.4$. The Lorenz system is a bounded chaotic attractor, i.e., there exists a constant M satisfying$\|f(x_i(t)) - f(\bar{x}(t))\| \leq 2M\|e_i(t)\|$. $x_0 = 20*\ (1, 0.5, -1, 1.2, 0.6, -1.2,\ 1.4, 0.7, -1.4, 1.6, 0.8,\ -1.6, 1.8,\ 0.9, -1.8, 2, 1, -2, -1.8, 1.1, 1.8,\ -1.6, 1.2,\ 1.6, -1.4, 1.3, 1.4,\ -1.2, 1.4,\ 1.2, -1, 1.5, 1, -3, -2.5, -2, -1.5, -1, -0.5, -0.5, -1, -1.5, -2, -2.5)^T$. Adaptive gains $(h_1, h_2, \cdots, h_{11}) = (0.3, 0.4, 0.5, 0.6, 0.3, 0.4, 0.5, 0.6, 0.3, 0.4, 0.5)$. Applying Theorem 2.2 we know that the exact consensus is achieved. Simulation results are depicted in Figure 2.1 to Figure 2.5 respectively

2.6 Conclusions

This chapter has studied the exact consensus problem for a NMAS with nonlinear agent dynamics and communication delay. We use the average dynamics of all agents as the desired moving trajectories, then we have presented linear feedback

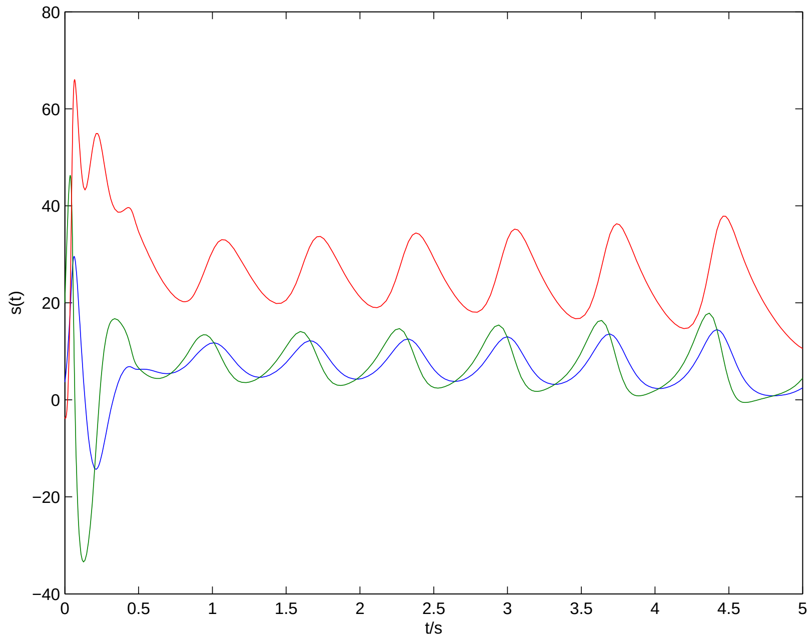

Figure 2.1: Desired Moving Trajectories

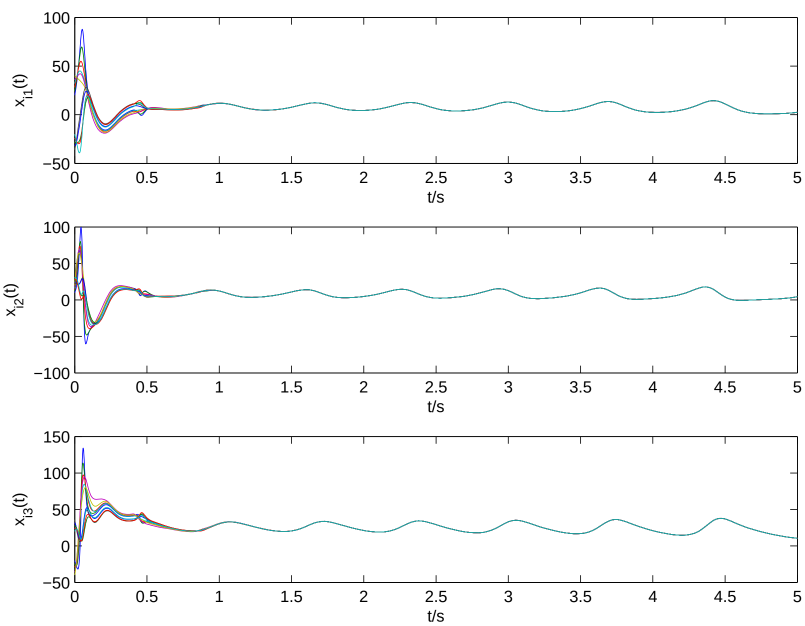

Figure 2.2: Agent Dynamics

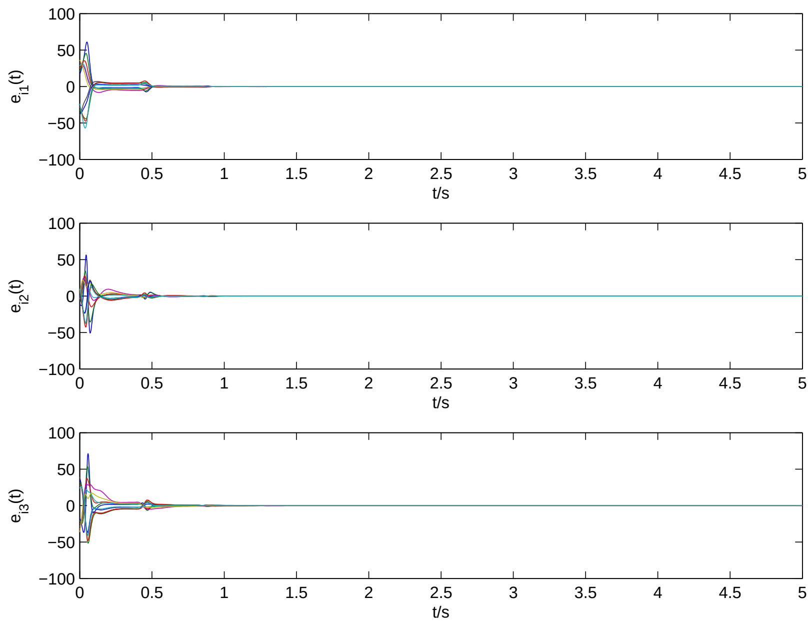

Figure 2.3: Error System Dynamics

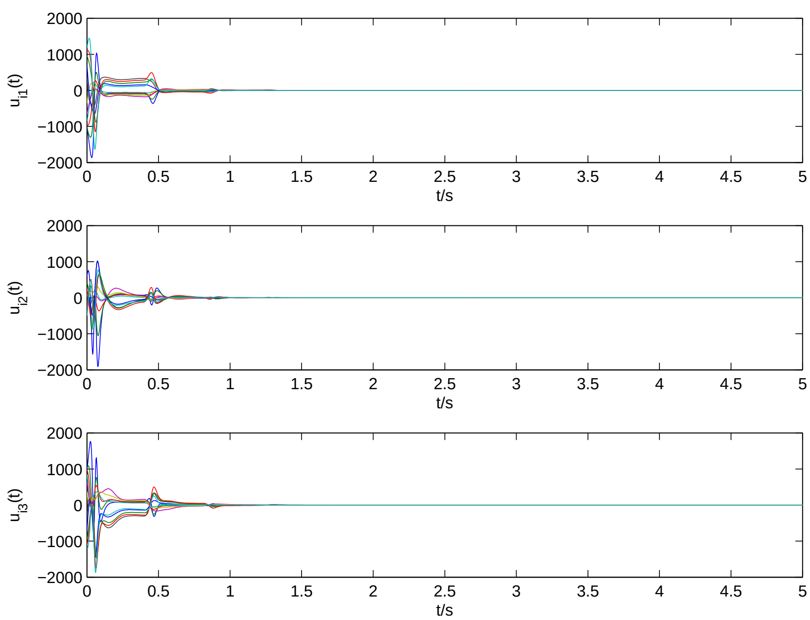

Figure 2.4: Adaptive Controllers

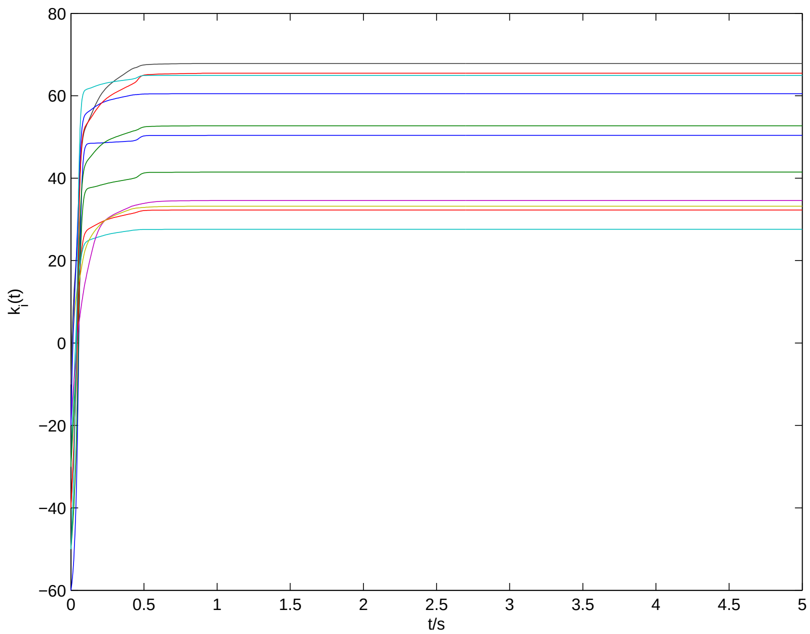

Figure 2.5: Adaptive Gain Curves

and adaptive feedback to guarantee its global exact consensus based on the Lyapunov stability theory. The controllers are very simple in form, but are more effective to resolve the consensus problem of the NMAS with nonlinear agent dynamics and communication delay. The simulation results have illustrated the effectiveness of the proposed results. It should be noted that the conditions are still restrictive and all the delays are the same, further work will focus on these problems.

Chapter 3

Controlled Consensus Criteria for NMAS with Uncertain Coupling

This chapter investigates the consensus problem for uncertain NMAS. Based on the network connection topology and Lyapunov stability theory, we investigate the local and global decentralized consensus for such systems. Under the different assumptions, several consensus criteria are deduced, moreover, the assumptions adopted and decentralized control laws designed are considerably simple. Examples along with the respective numerical and computer simulation results are also given to demonstrate the effectiveness of the proposed consensus criterion.

3.1 Introduction

It is noted that the agent dynamics in most existing works are often restricted to linear and identical systems. Obviously, in practice, this is not always the case. The consensus problem of NMAS with non-identical agent dynamics is much more complicated than the identical case and some results have been reported to date, but these results are obtained under the assumption that the network typology is known in advance. Obviously, it is often difficult to determine the exact topology structure of the NMAS, instead, limited information is involved. It is also noticed that in many existing results the final state of the NMAS, where the system will reach after achieving consensus, is usually not known in advance and cannot be changed by design. Therefore, it is interesting to study a general situation where the agent network coupling is unknown but limited to bounded linear/nonlinear functions, and the network structure is only partially known or completely unknown a priori. The present chapter will focus on the global consensus problems of Multi-Agent System with uncertain coupling.

The rest of this chapter is organized as follows. A controlled NMAS model with uncertain coupling is presented in Section 3.2. The main results including decentralized state feedback control and decentralized output feedback control are derived in Section 3.3 and Section 3.4 respectively. In Section 3.5, numerical simulation examples are given to verify the effectiveness of the proposed results, followed by conclusions in Section 3.6.

3.2 Model Description

Consider an uncertain NMAS consisting of N identical nodes with diffusively coupling described by

$$\dot{x}_i(t) = f(x_i(t), t) + g_i(x_1(t), x_2(t), \cdots, x_N(t), t) + u_i, 1 \leq i \leq N, \tag{3.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f(x_i(t)) : R^n \to R^n$ are continuously differentiable mappings representing the self-dynamics of the agent v_i, $g_i \in R^n$ are unknown nonlinear smooth diffusive coupling functions, $u_i \in R^n$ are the control inputs, and the coupling-control terms satisfy $g_i(s(t), s(t), \cdots, s(t), t) + u_i = 0$ for all $t \geq 0$, where $s(t)$ is a consensus solution of the isolated agent system $\dot{x}(t) = f(x(t), t)$ and $s(t)$ can be an equilibrium point, a periodic orbit, an aperiodic orbit, or a chaotic orbit in the phase space.

The objective of the present paper is to achieve consensus for the uncertain Multi-Agent System (3.1) by designing appropriate controllers u_i. That is, the trajectories of the closed-loop systems satisfy:

$$\lim_{t\to\infty} \|x_i(t) - s(t)\|_2 = 0, \ 1 \leq i \leq N. \tag{3.2}$$

Define the error vector by $e_i(t) = x_i(t) - s(t)$; then the error dynamical system can be given as follows:

$$\dot{e}_i(t) = \bar{f}(x_i(t), s(t), t) + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) + u_i, \ 1 \leq i \leq N, \tag{3.3}$$

where $\bar{f}(x_i(t), s(t), t) = f(x_i(t), t) - f(s(t), t)$, $\bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) = g_i(x_1(t), x_2(t), \cdots, x_N(t), t) - g(s(t), s(t), \cdots, s(t), t)$.

3.3 Decentralized State Feedback Consensus

Linearizing error system (3.3) around zero gives

$$\dot{e}_i(t) = A(t)e_i(t) + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) + u_i,\ 1 \leq i \leq N, \tag{3.4}$$

where $A(t) = Df(s(t), t)$ is the Jacobian of f evaluated at $s(t)$.
Obviously, it follows the pair $(A(t), I)$ is controllable. Then there exist matrices $K(t)$, $P(t) > 0$ and $Q(t) > 0$ such that

$$\dot{P}(t) = -(A(t) + K(t))^T P(t) - P(t)(A(t) + K(t)) - Q(t). \tag{3.5}$$

To achieve the objective (3.2), we need the following assumptions.
Assumption 3.1 (A1). Suppose that there exist known first-order continuously differentiable positive definite functions $\varphi_i(.)$ with $\varphi_i(0) = 0$ and nonnegative functions $r_{ij}(t)$ such that

$$\|\bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t))\| \leq \sum_{j=1}^{N} r_{ij}(t)\varphi_i(\|e_j(t)\|), \quad 1 \leq i \leq N, \tag{3.6}$$

for $x(t) \in \Omega, t \in R^+$.
A local decentralized consensus criterion is deduced as follows.
Theorem 3.1. Suppose there exists a neighborhood about origin $\bar{\Omega} \subseteq \Omega$ such that the matrix function $W^T(e(t)) + W(e(t)) > 0$ in $\bar{\Omega}\backslash\{0\}$, where $W(e(t)) = (w_{ij}(e(t)))_{N\times N}$ is defined by

$$w_{ij}(e(t)) = \begin{cases} \lambda_m(Q(t)) - 2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i = j, \\ -2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i \neq j, \end{cases}$$

where $\lambda_m(.)$ and $\lambda_M(.)$ stand for the smallest and largest eigenvalues respectively,

$\phi_i(r) = \int_0^1 \frac{\partial \varphi_i(r\zeta)}{\partial \zeta} d\zeta$ with $r \in R^+$. If A1 is also satisfied, then the consensus solution $S(t)$ of the uncertain NMAS (3.1) is locally asymptotically stable under the decentralized controllers

$$u_i = K(t)e_i(t), \quad 1 \leq i \leq N. \tag{3.7}$$

Proof: Substituting (3.7) into (3.4) gives the following closed-loop error system

$$\dot{e}_i(t) = (A(t) + K(t))e_i(t) + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t),\ 1 \leq i \leq N. \tag{3.8}$$

Select the following Lyapunov function candidate

$$V(e(t)) = \sum_{i=1}^{N} e_i^T(t)P(t)e_i(t), \tag{3.9}$$

where $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$ and $P(t)$ is defined by (3.5). The time derivative of $V(e(t))$ along the solution of the closed-loop error system (3.8) is

$$\begin{aligned} \dot{V}(e(t)) &= \sum_{i=1}^{N} \dot{e}_i^T(t)P(t)e_i(t) + e_i^T(t)\dot{P}(t)e_i(t) + e_i^T(t)P(t)\dot{e}_i(t) \\ &= \sum_{i=1}^{N} e_i^T(t)((A(t) + K(t))^T P(t) + P(t)(A(t) + K(t)) + \dot{P}(t))e_i(t) \\ &+ 2\sum_{i=1}^{N} e_i^T(t)P(t)\bar{g}_i(x_1(t), x_2(t), \cdots, x_{N(t)}, s(t), t). \end{aligned} \tag{3.10}$$

By virtue of A1 and (3.13), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} e_i^T(t)((A(t) + K(t))^T P(t) + P(t)(A(t) + K(t)) + \dot{P}(t))e_i(t) \\ &\qquad \leq -\sum_{i=1}^{N} e_i^T(t)Q(t)e_i(t). \end{aligned}$$

$$2\sum_{i=1}^{N} e_i^T(t)P(t)\bar{g}_i(x_1(t), x_2(t), \cdots, x_{N(t)}, s(t), t)$$
$$\leq 2\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|)\|e_i(t)\|\|e_j(t)\|.$$

Now, substituting (3.11) and (3.11) into (3.10) yields

$$\begin{aligned}
\dot{V}(e(t)) \leq & -\sum_{i=1}^{N} \lambda_m(Q(t))\|e_i(t)\|^2 \\
& + 2\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|)\|e_i(t)\|\|e_j(t)\| \\
& = -\frac{1}{2}(\|e_1(t)\|, \|e_2(t)\|, \cdots, \|e_N(t)\|)(W^T(e(t)) + W(e(t))) \\
& (\|e_1(t)\|, \|e_2(t)\|, \cdots, \|e_N(t)\|)^T.
\end{aligned}$$

From the positive definitiveness of function matrix $W^T(e(t))+W(e(t))$ in $\Omega\setminus\{0\}$, it follows that $\dot{V}(e(t))$ is a negative definite function in domain Ω. Therefore, the error dynamical system (3.4) is locally asymptotically stabilized by the controllers (3.7), i.e., $\lim_{t\to\infty} \|x_i(t) - s(t)\|_2 = 0,\ 1 \leq i \leq N$. Consequently the consensus solution $S(t)$ of the uncertain NMAS (3.1) is locally asymptotically stable under the decentralized controllers (3.7). The proof is thus completed.

Remark 3.1 Compared with many existing results on the consensus of the NMAS, the above result generalizes the average liner coupling to nonlinear coupling and the proposed controllers are very simple in form. The conformance of the NMAS may be reinforced when the controllers are associated with more agent information, but the coupling density of the Multi-Agent System increases greatly at the same time. In fact, the connections among nodes are very sparse in practice requiring a tradeoff between the conformance of the NMAS and the operating regions of the controllers.

As a special case, assume that the nonlinear coupling terms of the NMAS (3.1) are bounded by linear functions, that is to say, the inequalities (3.6) satisfy $\|\bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t)\| \leq \sum_{j=1}^{N} r_{ij}(t)\|e_j(t)\|$, $1 \leq i \leq N$, then one arrives at the following corollary.

Corollary 3.1 Suppose there exists a neighborhood about the origin $\bar{\Omega} \subseteq \Omega$ such that the matrix function $W^T(e(t)) + W(e(t)) > 0$ in $\bar{\Omega} \setminus \{0\}$, then the consensus solution $S(t)$ of the NMAS (3.1) with linear coupling is locally asymptotically stable under the decentralized set of controllers (3.7), where

$$w_{ij}(e(t)) = \begin{cases} \lambda_m(Q(t)) - 2\lambda_M(P(t))r_{ij}(t), & i = j, \\ -2\lambda_M(P(t))r_{ij}(t), & i \neq j. \end{cases}$$

Investigating the global decentralized consensus of the NMAS (3.1), we rewrite node dynamics $\dot{x}_i(t) = f(x_i(t), t)$ as $\dot{x}_i = A(t)x_i(t) + h(x_i(t), t)$, where $A(t) \in R^{n\times n}$ and $h : \Omega \times R^+ \to R^n$ is a smooth nonlinear function. The NMAS (3.1) is thus described by

$$\dot{x}_i(t) = A(t)x_i(t) + h(x_i(t), t) + g_i(x_1(t), x_2(t), \cdots, x_N(t), t) + u_i,\ 1 \leq i \leq N. \tag{3.11}$$

Similarly, the error system may be determined as:

$$\dot{e}_i(t) = A(t)e_i(t) + \bar{f}(x_i(t), s(t), t) + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) + u_i, \tag{3.12}$$

where $\bar{f}(x_i(t), s(t), t) = h(x_i(t), t) - h(s(t), t)$.

Assumption 3.2 (A2). Suppose that there exist known first-order continuously differentiable positive definite functions $\gamma_i(.)$ with $\gamma_i(0) = 0$ such that $\|\bar{f}(x_i(t), s(t), t)\| \leq \gamma_i(\|e_i(t)\|)$.

The global decentralized consensus criterion for the NMAS (3.1) is investigated below.

Theorem 3.2 Suppose that A1 and A2 are satisfied. If there exist a neighborhood about origin $\bar{\Omega} \subseteq \Omega$ such that the matrix function $W^T(e(t)) + W(e(t)) > 0$ in $\bar{\Omega}\backslash\{0\}$, where $W(e(t)) = (w_{ij}(e(t)))_{N\times N}$ is defined by

$$w_{ij} = \begin{cases} \lambda_m(Q(t)) - 2\lambda_M(P(t))\kappa_i(\|e_i(t)\|) - \lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i = j, \\ -\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i \neq j, \end{cases}$$

for $i, j = 1, 2, \cdots, N$, $\phi_i(r) = \int_0^1 \frac{\partial \varphi_i(r\zeta)}{\partial \zeta} d\zeta$ and $\kappa_i(r) = \int_0^1 \frac{\partial \gamma_i(r\zeta)}{\partial \zeta} d\zeta$ with $r \in R^+$. Then the consensus solution $S(t)$ of the uncertain NMAS (3.1) is globally asymptotically stable under the decentralized controllers (3.7).

The proof is very similar to that of the Theorem 4.1, and is omitted here.

Remark 3.2 In Theorem 3.2, nonlinear functions $\bar{f}(x_i(t), s(t), t)$ in the NMAS are bounded by known continuous differentiable functions and the proposed decentralized controllers (3.7) are associated with the bounds of the nonlinear functions, which leads to the conservativeness of the results. If more information of the system's nonlinearity is used and nonlinear controllers are designed, then the conservativeness of the results can be reduced [92, 94].

Theorem 3.3 Suppose $\bar{f}(x_i(t), s(t), t) = 0$ for $e_i(t) \in \Xi, \Xi = \{(e_i(t), t)|P(t)e_i(t) = 0, t \in R^+\}$ for $1 \leq i \leq N$. If A2 is satisfied and there exists a neighborhood about origin $\bar{\Omega} \subseteq \Omega$ such that the matrix function $W^T(e(t)) + W(e(t)) > 0$ in $\bar{\Omega}\backslash\{0\}$, where $W(e(t)) = (w_{ij}(e(t)))_{N\times N}$ is defined by

$$w_{ij}(e(t)) = \begin{cases} \lambda_m(Q(t)) - 2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i = j, \\ -2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i \neq j, \end{cases}$$

where $\lambda_m(.)$ and $\lambda_M(.)$ represent the smallest and largest eigenvalues respectively, $\phi_i(r) = \int_0^1 \frac{\partial \varphi_i(r\zeta)}{\partial \zeta} d\zeta$ with $r \in R^+$, then the consensus solution $S(t)$ of the uncertain NMAS (3.1) is globally asymptotically stable under the decentralized controllers

$$u_i = K(t)e_i(t) + \rho(e_i(t)), \quad 1 \leq i \leq N, \tag{3.13}$$

where $\rho(.)$ is given by

$$\rho(e_i(t) = \begin{cases} -\frac{P(t)e_i(t)}{\|P(t)e_i(t)\|^2} \lambda_M(P(t))\gamma_i(\|e_i(t)\|)\|e_i(t)\|, \ P(t)e_i(t) \neq 0, \\ 0, \ P(t)e_i(t) = 0. \end{cases} \tag{3.14}$$

The proof is very similar to that of the Theorem 3.1, and is omitted here.

3.4 Decentralized Output Feedback Consensus

Now we consider the following controlled uncertain Multi-Agent System with outputs.

$$\begin{cases} \dot{x}_i(t) & = A(t)x_i(t) + h(x_i(t), t) + g_i(x_1(t), x_2(t), \cdots, x_N(t), t) + u_i, \\ y_i(t) & = H(t)x_i(t), \ i = 1, 2, \cdots, N, \end{cases} \tag{3.15}$$

where $y_i(t) \in R^n$ is the output of the $i-$th node, $H(t) \in R^{n \times n}$ and other statements on the NMAS are the same as in (3.1) and (3.11).

Similarly, one can get the following error system

$$\begin{cases} \dot{e}_i(t) & = A(t)e_i(t) + \bar{f}(x_i(t), s(t), t) + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) + u_i, \\ \bar{y}_i(t) & = H(t)e_i(t) + H(t)s(t), \ i = 1, 2, \cdots, N, \end{cases} \tag{3.16}$$

where $\bar{f}(x_i(t), s(t), t) = h(x_i(t), t) - h(s(t), t)$.

Then the object is to achieve consensus of the NMAS (3.15) by designing decentralized output feedback controllers $u_i(y_i(t))$, that is, the trajectories of the closed-loop systems satisfy (3.2). As usual, the consensus problem of the Multi-Agent System (3.15) is equivalent to the stabilization problem of the error dynamical system (3.16). To achieve the objective, we need the following assumptions.

Assumption 3.3 (A3). Suppose that there exist known first-order continuously differentiable positive definite functions $r_i(.)$ with $r_i(0) = 0$ such that $\|\bar{f}(x_i(t), s(t), t)\| \leq \bar{\gamma}_i(t)\|\bar{y}_i(t)\|$.

Theorem 3.4 Suppose that A1 and A3 are satisfied. If there exist matrix $K(t) \in R^{n\times n}$, nonsingular matrix $D(t) \in R^{n\times n}$, two symmetric positive definite matrices $P(t) \in R^{n\times n},\ Q(t) \in R^{n\times n}$, a neighborhood about origin $\bar{\Omega} \subseteq \Omega$ and a positive constant ϵ_1 such that $(A(t)+K(t)H(t))^T P(t)+P(t)(A(t)+K(t)H(t))+\dot{P}(t) = -Q(t)$, $P(t) = D(t)H(t)$ and $\epsilon_2 \geq \frac{\bar{\gamma}_i^2(t)}{\epsilon_1}\|P(t)\|^2$, and the matrix function $W^T(e(t))+W(e(t)) > 0$ in $\bar{\Omega}\backslash\{0\}$, where $W(e(t)) = (w_{ij}(e(t)))_{N\times N}$ is defined by

$$w_{ij}(e(t)) = \begin{cases} \lambda_m(Q(t)) - \epsilon_1 - \frac{1}{\epsilon_2}\|(K^T(t)D(t) - \frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}\|P(t)\|^2 I_n)\|^2\|H(t)\|^2 \\ \qquad -2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), \quad i = j, \\ -2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), \quad i \neq j, \end{cases}$$

for $r_{ij}(t)$, $\lambda_m(.)$ and $\lambda_M(.)$ represent the smallest and largest eigenvalues respectively, $\phi_i(r) = \int_0^1 \frac{\partial\varphi_i(r\zeta)}{\partial\zeta}d\zeta$ with $r \in R^+$. Then the consensus solution $S(t)$ of the uncertain Multi-Agent System (3.15) is globally asymptotically stable under the decentralized output feedback controllers

$$u_i(\bar{y}_i(t)) = u_i^a(\bar{y}_i(t)) + u_i^b(\bar{y}_i(t)), \tag{3.17}$$

where $u_i^a(\bar{y}_i(t)) = K(t)\bar{y}_i(t),\ u_i^b(\bar{y}_i(t)) = -\frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}(D^T(t))^{-1}\|P(t)\|^2\bar{y}_i(t)$. Proof. Substituting the controllers (3.17) into the error dynamical system (3.16), the closed-loop system of (3.16) is obtained as

$$\begin{cases} \dot{e}_i(t) &= (A(t)+K(t))e_i(t)+\bar{f}(x_i(t),s(t),t) \\ & \qquad +\bar{g}_i(x_1(t),x_2(t),\cdots,x_N(t),s(t),t)+u_i^b(\bar{y}_i(t)), \\ \bar{y}_i(t) &= H(t)e_i(t)+H(t)s(t),\ i=1,2,\cdots,N. \end{cases}$$

Define a Lyapunov function candidate as follows:

$$V(e(t)) = \sum_{i=1}^{N} e_i^T(t)P(t)e_i(t). \tag{3.18}$$

The derivative of V(e(t)) along the trajectory of (3.18) is given by

$$\begin{aligned} \dot{V}(e(t)) = \sum_{i=1}^{N} & -e_i^T(t)Q(t)e_i(t) \\ & + 2\sum_{i=1}^{N} e_i^T(t)P(t)(\bar{f}(x_i(t),s(t),t)+K(t)H(t)s(t)+u_i^b(\bar{y}_i(t))) \\ & + 2\sum_{i=1}^{N} e_i^T(t)P(t)\bar{g}(x_1(t),x_2(t),\cdots,x_N(t),s(t),t). \end{aligned}$$

From the conditions of the theorem and the definition of $u_i^b(\bar{y}_i(t))$, we obtain

$$\begin{aligned} & 2\sum_{i=1}^{N}(e_i^T(t)P(t)(\bar{f}(e_i(t),s(t),t)+K(t)H(t)s(t)+u_i^b(\bar{y}_i(t)))) \\ & = \sum_{i=1}^{N}(2e_i^T(t)P(t)\bar{f}(x_i(t),s(t),t)+2e_i^T(t)P(t)u_i^b(\bar{y}_i(t)) \\ & \quad + 2e_i^T(t)P(t)K(t)H(t)s(t)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N}(2e_i^T(t)P(t)\bar{f}(e_i(t), s(t), t) + 2(e_i(t) + s(t))^T H^T(t)D^T(t)u_i^b(\bar{y}_i(t)) \\
&\quad - 2s^T(t)H^T(t)D^T(t)u_i^b(\bar{y}_i(t)) + 2(e_i(t) + s(t))^T P(t)K(t)H(t)s(t) \\
&\quad - 2s^T(t)P(t)K(t)H(t)s(t))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N}(2e_i^T(t)P(t)\bar{f}(e_i(t), s(t), t) + 2(D(t)\bar{y}_i(t))^T u_i^b(\bar{y}_i(t)) \\
&\quad - 2s^T(t)H^T(t)D^T(t)u_i^b(\bar{y}_i(t)) \\
&\quad + 2(D(t)\bar{y}_i(t))^T K(t)H(t)s(t) - 2s^T(t)P(t)K(t)H(t)s(t)) \\
&\leq \sum_{i=1}^{N}(2\|e_i(t)\|\|P(t)\|\bar{\gamma}_i(t)\|\bar{y}_i(t)\| + 2(D(t)\bar{y}_i(t))^T u_i^b(\bar{y}_i(t)) \\
&\quad - 2s^T(t)H^T(t)D^T(t)u_i^b(\bar{y}_i(t)) \\
&\quad + 2(D(t)\bar{y}_i(t))^T K(t)H(t)s(t) - 2(H(t)s(t))^T D^T(t)K(t)(H(t)s(t)))
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{N}(\epsilon_1\|e_i(t)\|^2 + \frac{\bar{\gamma}_i^2(t)}{\epsilon_1}\|P(t)\|^2\|\bar{y}_i(t)\|^2 \\
&\quad + 2(D(t)\bar{y}_i(t))^T(-\frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}(D^T(t))^{-1}\|P(t)\|^2\bar{y}_i(t)) \\
&\quad + 2s^T(t)H(t)^T D^T(t)(-\frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}(D^T(t))^{-1}\|P(t)\|^2\bar{y}_i(t)) \\
&\quad + 2(D(t)\bar{y}_i(t))^T K(t)H(t)s(t) - 2(H(t)s(t))^T D^T(t)K(t)(H(t)s(t)))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N}(\epsilon_1\|e_i(t)\|^2 + 2s^T(t)H^T(t)(K^T(t)D(t) - \frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}\|P(t)\|^2 I_n)\bar{y}_i(t) \\
&\quad - 2(H(t)s(t))^T D^T(t)K(t)(H(t)s(t)))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N}(\epsilon_1\|e_i(t)\|^2 + 2s^T(t)H^T(t)(K^T(t)D(t) - \frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}\|P(t)\|^2 I_n)H(t)e_i(t) \\
&\quad - (H(t)s(t))^T(\frac{\bar{\gamma}_i^2(t)}{\epsilon_1}\|P(t)\|^2 I_n)(H(t)s(t)))
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{N}(\epsilon_1\|e_i(t)\|^2 \\
&\quad + 2\|H(t)s(t)\|\|(K^T(t)D(t) - \frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}\|P(t)\|^2 I_n)\|\|H(t)\|\|e_i(t)\| \\
&\quad - (H(t)s(t))^T(\frac{\bar{\gamma}_i^2(t)}{\epsilon_1}\|P(t)\|^2 I_n)(H(t)s(t))) \\
&\leq \sum_{i=1}^{N}(\epsilon_1\|e_i(t)\|^2 + \epsilon_2\|H(t)s(t)\|^2 \\
&\quad + \frac{1}{\epsilon_2}\|(K^T(t)D(t) - \frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}\|P(t)\|^2 I_n)\|^2\|H(t)\|^2\|e_i(t)\|^2 \\
&\quad - (H(t)s(t))^T(\frac{\bar{\gamma}_i^2(t)}{\epsilon_1}\|P(t)\|^2 I_n)(H(t)s(t))) \\
&\leq \sum_{i=1}^{N}((\epsilon_1 + \frac{1}{\epsilon_2}\|(K^T(t)D(t) - \frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}\|P(t)\|^2 I_n)\|^2\|H(t)\|^2)\|e_i(t)\|^2),
\end{aligned}
$$

and

$$
\begin{aligned}
2\sum_{i=1}^{N} & e_i^T(t)P(t)\bar{g}_i(x_1(t), x_2(t), \cdots, x_{N(t)}, s(t), t) \\
&\leq 2\sum_{i=1}^{N}\|e_i(t)\|\|P(t)\|(\sum_{j=1}^{N} r_{ij}(t)\varphi_i(\|e_j(t)\|)) \\
&\leq 2\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|)\|e_i(t)\|\|e_j(t)\|.
\end{aligned} \tag{3.19}
$$

By substituting (3.4) and (3.19) into (3.4), one gets

$$
\begin{aligned}
\dot{V}(e(t)) \leq -\frac{1}{2}&(\|e_1(t)\|, \|e_2(t)\|, \cdots, \|e_N(t)\|)(W^T(e(t)) + W(e(t))) \\
&(\|e_1(t)\|, \|e_2(t)\|, \cdots, \|e_N(t)\|)^T.
\end{aligned}
$$

From the positive definitiveness of function matrix $W^T(e(t))+W(e(t))$ in $\Omega\setminus\{0\}$, it follows that $\dot{V}(e(t))$ is a negative definite function in domain Ω. Therefore, the error dynamical system (3.18) is globally asymptotically stabilized by the controllers (3.17), i.e., $\lim_{t\to\infty} \|x_i(t) - s(t)\|_2 = 0,\ 1 \le i \le N$. Consequently the consensus solution $S(t)$ of uncertain NMAS (3.15) is locally asymptotically stable under the decentralized controllers (3.17). The proof is thus completed.

Remark 3.3 The above consensus criterion is associated with some time-varying parameters, which result in the difficulty of computation. As a special case, we assume that all time-varying parameters in the conditions of Theorem 3.4 are invariable for $t > 0$, then we can get the following corollary.

Corollary 3.2 Suppose that all time-varying parameters in the conditions of Theorem 3.4 are invariable for $t > 0$, then the consensus solution $S(t)$ of the uncertain NMAS (3.15) is globally asymptotically stable under the decentralized output feedback controllers $u_i(\bar{y}_i(t)) = (K - \frac{\bar{\gamma}_i^2}{2\epsilon_1}(D^T)^{-1}\lambda_M^2(P))\bar{y}_i(t)$, where

$$w_{ij} = \begin{cases} \lambda_m(Q) - \epsilon_1 - \frac{1}{\epsilon_2}\|K^T D - \frac{\bar{r}_i^2}{2\epsilon_1}\lambda_M^2(P)I_n\|^2\lambda_M(H) \\ \qquad -2\lambda_M(P)r_{ij}\phi_i(\|e_j(t)\|), \quad i = j, \\ -2\lambda_M(P)r_{ij}\phi_i(\|e_j(t)\|), \quad i \neq j, \end{cases}$$

$K, D, P, Q, \epsilon_1, \epsilon_2, \gamma_i$ and r_{ij} are corresponding constant parameters in Theorem 3.4.

Remark 3.4 Corollary 3.2 is the special case of the Theorem 3.4 and the conditions here are relatively easier to be obtained and verified. The assumptions and decentralized control laws are considerably simple compared with many existing results.

3.5 Examples

In this section, several numerical simulations for verifying the effectiveness of the proposed consensus criteria are provided.

Example 1. Consider the NMAS consisting of 2 identical second-order agents, which is described by

$$\begin{pmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} + 0.05sin(t)e^{-t} \begin{pmatrix} cos^2(x_{12}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$
$$+ sin(t)e^{-2t} \begin{pmatrix} sin(x_{12}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} + \begin{pmatrix} sin^2(x_{11})x_{12} \\ cos(x_{12})x_{11} \end{pmatrix} + u_1,$$
$$\begin{pmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} + e^{-t} \begin{pmatrix} 1 & 0 \\ 0 & sin(x_{21}^2) \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$
$$+ sin(t)e^{-2t} \begin{pmatrix} cos(x_{12}) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} + \begin{pmatrix} sin^2(x_{21})x_{22} \\ cos(x_{22})x_{21} \end{pmatrix} + u_2.$$

In the simulation, the controller gain matrix and other parameter matrix are chosen as follows

$$K(t) = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}, \quad P(t) = \begin{pmatrix} 0.6406 & 0.2031 \\ 0.2031 & 0.5156 \end{pmatrix}.$$

By computing directly, we have that

$$Q(t) = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \ W(t) = \begin{pmatrix} 4.42 - 0.04|sint|e^{-t} & -0.79|cos(t)|e^{-2t} \\ -0.79e^{-t} & 4.42 - 0.79|sin(t)|e^{-2t} \end{pmatrix},$$

$\|\bar{g}_1(x_1, s, t)\|_2 \le \|e_1\|_2$, $\|\bar{g}_2(x_2, s, t)\|_2 \le \|e_2\|_2$, $\|\bar{h}_1(x_1, x_2, s, t)\|_2 \le 0.05e^{-t}|sin(t)|$ $\|e_1\|_2 + e^{-2t}|cos(t)|\|e_2\|_2$, and $\|\bar{h}_2(x_1, x_2, s, t)\|_2 \le e^{-t}\|e_1\|_2 + e^{-2t}|sin(t)|\|e_2\|_2$. It is observed that the conditions of the Theorem 3.2 are satisfied. Choose the initial state as $x_0 = (-0.5, 0.7, 0.45, -0.3)$ and the consensus error e_i and the corresponding control signal are shown in Figure 3.1.

Example 2: Consider the following controlled time-varying NMAS consisting of

2 identical second-order nodes.

$$\begin{pmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{pmatrix} = \begin{pmatrix} -x_{12} \\ -sin(-x_{11}+2x_{12}) \end{pmatrix} - 0.04sin(t) \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$
$$+ 0.04sin(t) \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1,$$

$$y_1 = x_{11} - 2x_{12},$$

$$\begin{pmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{pmatrix} = \begin{pmatrix} -x_{22} \\ -sin(-x_{21}+2x_{22}) \end{pmatrix} + 0.03sin(t) \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$
$$- 0.03sin(t) \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2,$$

$$y_2 = x_{21} - 2x_{22}.$$

Let $u_1 = sin(-x_{11}+2x_{12}) + x_{11} - 2x_{12} = sin(y_1) - y_1$, $u_2 = sin(-x_{21}+2x_{22}) - x_{21} + 2x_{22} = sin(y_2) - y_2$, $V(z_1, z_2) = 3z_1^2 - 2z_1z_2 + z_2^2$. A direct computation gives: $r_{11}(\tau) = r_{21}(\tau) = (2-\sqrt{2})\tau^2$, $r_{21}(\tau) = r_{22}(\tau) = (2-\sqrt{2})\tau^2$, $k_1 = 3$, $k_2 = 7$,

$$W(t) = \begin{pmatrix} 3 - 0.28|sin(t)|e^{-t} & 0.28|sin(t)|e^{-t} \\ 0.21|cos(t)|e^{-t} & 3 - 0.21|cos(t)|e^{-t} \end{pmatrix}$$

is positive definite for any $t > 0$. The conditions of the Theorem 3.4 are satisfied. Therefore, the above network can be stabilized by the decentralized output feedback controller. Choose the initial state as $x_0 = (-0.5, 0.7, 0.45, -0.3)$ and the simulation results are depicted in Figure 3.2.

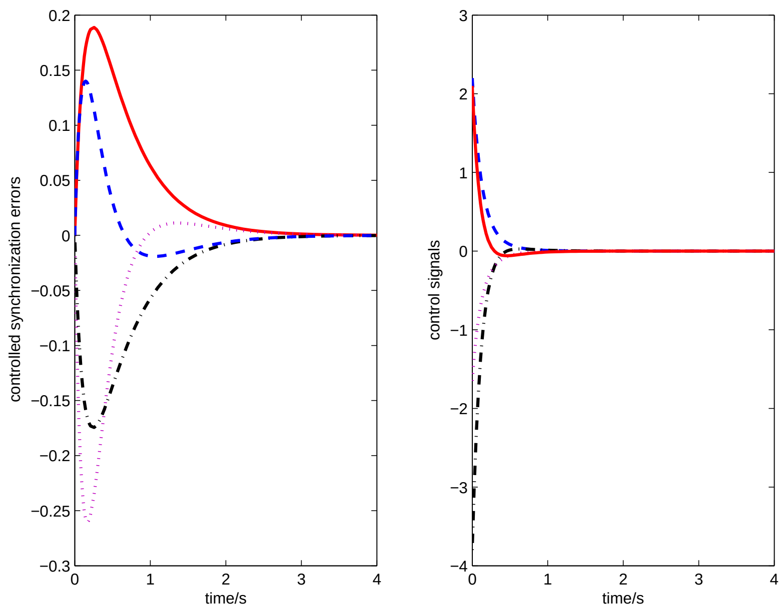

Figure 3.1: Consensus errors and control signals.

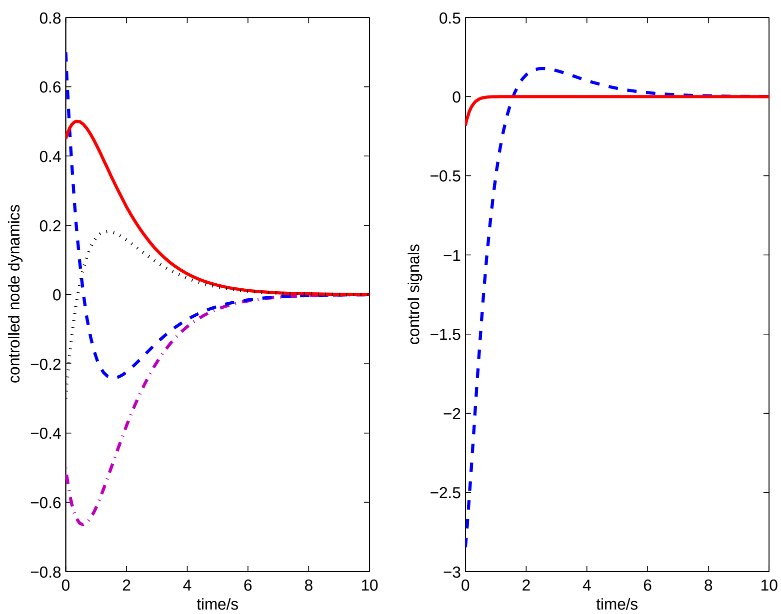

Figure 3.2: Consensus errors and control signals.

3.6 Conclusions

This chapter investigated the local and global consensus problems of NMAS with uncertain coupling structure. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the MAS is achieved based on a series of transformations and Lyapunov stability theorem. The controllers are very simple in form, but are more effective in resolving the consensus problem with uncertain coupling structure.

Part B

Global State Consensus of MAS with Different Agent Dynamics and Communication Delay

Chapter 4

Controlled Consensus of NMAS with Different Agent Dynamics

The global consensus problem of NMAS with different agent dynamics will be investigated in this chapter. The proposed consensus property is formulated in terms of certain boundedness of state errors and the exact consensus is achieved by using nonlinear controllers. Based on Lyapunov stability theorem, the bounded and exact consensus criterion has been proved systematically. At last, the main results are illustrated by numerical simulations and the simulation results demonstrate the effectiveness of the proposed methods.

4.1 Introduction

The consensus problem means to reach an agreement that depends on the states of all agents and the topic has been studied across many fields of science and engineering. Most of the existing results focus on the NMAS with liner and identical agent dynamics instead of nonlinear and non-identical agent dynamics because the latter is much more complicated than the former.

This chapter will focus on the global consensus problems of NMAS and the proposed consensus criterion is formulated in terms of certain boundedness of state errors, in addition, the exact consensus are also investigated by means of nonlinear controllers. The remainder part of this chapter is organized as follows. A continuous-time NMAS model with non-identical agent dynamics is presented in Section 4.2. The main results including bounded consensus and exact consensus criterion are derived in Section 4.3, 4.4 and 4.5. In Section 4.6, numerical simulation examples are given to verify the effectiveness of the proposed results, followed by conclusions in Section 4.7.

4.2 Problem Description

Consider a NMAS consisting of N non-identical agent dynamics with communication delay:

$$\dot{x}_i(t) = B_i x_i(t) + c \sum_{j \in \mathcal{N}_i}^{N} a_{ij} \Gamma x_j(t-\tau) + u_i, \quad i = 1, 2, \cdots, N, \tag{4.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $B_i \in R^{n \times n}$ are constant matrices, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n \times n}$ is the inner coupling matrix, which is a zero-one constant matrix, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N\times N}$, which is symmetric and irreducible, representing the communication topology relation of the NMAS, is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j\neq i} a_{ij}$. τ is a constant coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The desired moving trajectory is chosen as

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} B_k x_k(t). \tag{4.2}$$

We'll now discuss the problem of global consensus for the NMAS (4.1). The consensus problem formulation here is quite different from many existing results, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. The consensus problem here will be depicted instead via certain boundedness of $x_i(t) - x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. At the same time, we will design controllers for the NMAS (4.1) to guarantee it achieve exact consensus. To address these cases we will focus on making the states of all agents converge to a bounded set or an equilibrium point.

Before stating the main results of this paper, the following mathematical preliminaries are necessary.

Definition 4.1([92, 93]): The solution $x_i(t, t_0, \psi_i)$ of the MAS (4.1) is said to be uniformly ultimately bounded with respect to the bound ε if for each $\delta > 0$ there exists $T = T(\varepsilon, \delta) > 0$ independent of t_0 such that $\|x_i(t, t_0, \psi_i)\| \leq \varepsilon$ for all $t \geq t_0 + T$ when $\|x_i(t_0)\| < \delta$, where $\psi_i(t)$ is the initial value given as $x_i(t) = \psi_i$ for $t \in [t_0 - \tau, t_0]$, $i = 1, 2, \cdots, N$.

Lemma 4.1([88]): Assuming that the graph $G = (\mathcal{V}, \mathcal{A})$ is a strongly connected graph, then there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \cdots, \Phi_N)$, such that the adjacency matrix A satisfies

$$\Phi^T A \Phi = \Lambda = diag\{\lambda_1, \lambda_2, \cdots, \lambda_N\}, \tag{4.3}$$

where Φ_i is the i-th column of Φ with $\Phi_1 = (\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}})^T$ and $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$ are the eigenvalues of A.

Lemma 4.2 ([88]): Let $g(t)$ be a non-negative bounded function defined on R^+ and

$$\Omega = \{x(t) \in R^n | \|x(t)\| \leq \overline{lim}_{t\to\infty} g(t)\}. \tag{4.4}$$

Suppose there exist a strictly positive definite matrix $P(t) \in \mathcal{PC}^1_{n\times n}$ and a constant $\delta > 0$ such that the derivative of $V(x(t), t) = x^T(t)P(t)x(t)$ along the trajectory of the system

$$\dot{x}(t) = f(x(t), t), \quad x(t) \in R^n, t \in [0, \infty) \tag{4.5}$$

satisfies

$$\dot{V} \leq -\delta \|x(t)\|^2 \;\; \texttt{if} \;\; \|x(t)\| \geq g(t). \tag{4.6}$$

For any $t > 0$, let

$$Q_t = \{x(t) | V(x(t), t) \leq \sup_{y(s)\in\Omega, s\geq 0} \{V(y(s), s)\}\} \tag{4.7}$$

and

$$c = \overline{lim}_{t\to\infty}(max\{\|x(t)\| | x(t) \in Q_t\}). \tag{4.8}$$

Then, $x(t)$ converges to the set

$$M = \{x(t) | \|x(t)\| \leq c\}. \tag{4.9}$$

We now introduce some notations and definitions.

Let $PC^1_{n\times n}$ be the linear space of the uniformly bounded continuous real matrix-valued functions defined on $[0, \infty)$. For any $P \in PC_{n\times n}$ the norm of P is defined by $\|P\| = \max\limits_{0\leq t<\infty} \{\|P(t)\|\}$.

By defining the consensus error vector as

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \cdots, N. \tag{4.10}$$

Obviously, $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i}^{N} a_{kj}\Gamma x_j(t-\tau) = 0$, from (4.1) and (4.2), the error dynamics are found to be

$$\dot{e}_i(t) = B_i x_i(t) - \frac{1}{N}\sum_{k=1}^{N} B_k x_k(t) + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma x_j(t-\tau) \tag{4.11}$$

for $i = 1, 2, \cdots, N$. Assuming that the trajectories of all agents are bounded, it follows that their difference is also bounded. Then, the following inequalities can be defined

$$\|B_i x_i(t) - \frac{1}{N}\sum_{k=1}^{N} B_k x_k(t)\| \leq \alpha_i \tag{4.12}$$

for $i = 1, 2, \cdots, N$, where α_i are nonnegative constants.

In order to achieve consensus, the local controllers u_i have to be designed such that $e_i(t)$ becomes asymptotically stable about its zero fixed point. Here, we will consider the problems of consensus analysis, controlled bounded consensus and controlled exact consensus respectively.

4.3 Bounded Consensus Analysis

We firstly consider the problem of consensus analysis of the NMAS (4.1) and the main result of this contribution can be stated as follows:

Theorem 4.1. If $c > \frac{-1}{\lambda_i}(\lambda_i \neq 0)$ and $0 < \delta < 1$, then the NMAS(4.1) without time delay will achieve bounded consensus and the consensus set is depicted as follows

$$M = \{e(t) | \|e(t)\| \leq \frac{\sum_{j=1}^{N} \alpha_j \overline{lim}_{t\to\infty} \|w_{ji}(t)\|}{1-\delta}\}, \tag{4.13}$$

namely, $e_i(t) = x_i(t) - s(t) \to \Omega$ as $t \to \infty$, where λ_i are the eigenvalues of A.

Proof. Discarding the controllers and rewriting the error system (4.11) in vector form one gets:

$$\dot{\bar{e}}(t) = \bar{B}(t) + c\Gamma\bar{e}(t)A^T, \tag{4.14}$$

where $\bar{e}(t) = [e_1(t), e_2(t), \cdots, e_N(t)] \in R^{n\times N}$, $\bar{B}(t) = [\hat{B}_1(t), \hat{B}_2(t), \cdots, \hat{B}_N(t)] \in R^{n\times N}$, $\hat{B}_i(t) = B_i x_i(t) - \frac{1}{N}\sum_{k=1}^{N} B_k x_k(t)$, $i = 1, 2, \cdots, N$.

Given that the connectivity matrix satisfies Lemma 4.1, there are two matrices, $\Omega = (w_1, w_2, \cdots, w_N) \in R^{N\times N}$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_N) \in R^{N\times N}$ such that:

$$A = \Omega^T\Lambda\Omega, \tag{4.15}$$

where λ_i and w_i are the $i-$th eigenvalue and associated eigenvector of A respectively. With $\Omega^T\Omega = I_N$, the $N-$dimensional identity matrix.

Using a change of variables $\bar{\eta}(t) = \bar{e}(t)\Omega^T$, the error dynamics become

$$\dot{\bar{\eta}}(t) = \bar{B}(t)\Omega^T + c\Gamma\bar{\eta}(t)\Lambda,$$

where $\bar{\eta}(t) = (\eta_1(t), \eta_2(t), \cdots, \eta_N(t))$, with $\eta_i(t) = \bar{e}(t)w_i^* \in R^n$ and $w_i^* = [w_{1i}, w_{2i}, \cdots, w_{Ni}]^T \in R^{N\times 1}$, or equivalently,

$$\dot{\eta}_i(t) = \bar{B}(t)w_i^* + c\Gamma\eta_i(t)\lambda_i, \quad i = 1, 2, \cdots, N. \tag{4.16}$$

The stability of the error dynamics (4.14) around the zero fixed point can be determine using the Lyapunov candidate function:

$$V = \frac{1}{2}\sum_{i=1}^{N} \eta_i^T(t)\eta_i(t). \tag{4.17}$$

The time derivative of V along the trajectories of the error dynamics in (4.16) is given by

$$\dot{V} = \sum_{i=1}^{N}(\eta_i^T(t)\bar{B}(t)w_i^*(t) + \eta_i^T(t)c\lambda_i\Gamma\eta_i(t)).$$

Considering the bounds of each term of $\dot{V}$. From (4.12) one has the bound of the first term:

$$\|\bar{B}(t)w_i^*\| \leq \|\sum_{j=1}^{N} B_j(t)w_{ji}\| \leq \|\sum_{j=1}^{N} \alpha_j w_{ji}\|.$$

The second term is always semi-negative because $c > 0$, $\lambda_i \leq 0$ and it can be expressed as

$$c\lambda_i\|\eta_i(t)\|^T\Gamma\|\eta_i(t)\| \leq -\|\eta_i(t)\|^T\|\eta_i(t)\|$$

if we take $c > \frac{-1}{\lambda_i}(\lambda_i \neq 0)$.

From the above results the time derivative V is bounded by

$$\dot{V} \leq \sum_{i=1}^{N}(\|\eta_i(t)\|^T \sum_{j=1}^{N} \alpha_j \|w_{ji}\| - \|\eta_i(t)\|^T \|\eta_i(t)\|).$$

If we take $\frac{\sum_{j=1}^{N} \alpha_j \|w_{ji}\|}{\|\eta_i(t)\|} < \delta < 1$, then we have

$$\dot{V} \leq (\delta - 1)\|\eta_i(t)\|^2.$$

Applying Lemma 4.2 completes the proof. In consequence, the NMAS (4.1) achieves bounded consensus.

Now we will consider the controlled consensus problem of NMAS (4.1), and bounded consensus criterion and exact consensus criterion are given as follows respectively.

Corollary 4.1. If $c\varpi > \frac{-2}{\lambda_i}(\lambda_i \neq 0)$ and $0 < \delta < 1$, then the delayed MNAS(4.1) will achieve bounded consensus and the consensus set is depicted as follows

$$M = \{e(t) | \|e(t)\| \leq \frac{\sum_{j=1}^{N} \alpha_j \overline{lim}_{t\to\infty} \|w_{ji}(t)\|}{1-\delta}\}, \tag{4.18}$$

namely, $e_i(t) = x_i(t) - s(t) \to \Omega$ as $t \to \infty$, where λ_i are the eigenvalues of A, ϖ is defined as follows.

$$\varpi = \begin{cases} 1 & \texttt{if} \;\; sign(\eta_i(t)) = sign(\eta_i(t-\tau)), \\ 0 & \texttt{else}. \end{cases} \tag{4.19}$$

Proof. Constructing the following Lyapunov functional:

$$V = \frac{1}{2}\varpi \sum_{i=1}^{N} \eta_i^T(t)\eta_i(t) + \sum_{i=1}^{N} \int_{t-\tau}^{t} \eta_i^T(\alpha)\eta_i(\alpha) d\alpha. \tag{4.20}$$

Repeat the similar as the proof of Theorem 4.1, then completes the proof.

4.4 Bounded consensus via linear negative feedback

Theorem 4.2. If the local controllers u_i are constructed as

$$u_i = -ck\Gamma e_i(t),$$

$k > \lambda_i + \frac{1}{c}$ and $0 < \delta < 1$, then the MAS(4.1) will achieve bounded consensus and the consensus set is depicted as follows

$$M = \{e(t)|\|e(t)\| \leq \frac{\sum_{j=1}^{N} \alpha_j \overline{lim}_{t\to\infty} \|w_{ji}(t)\|}{1-\delta}\}, \tag{4.21}$$

namely, $e_i(t) = x_i(t) - s(t) \to \Omega$ as $t \to \infty$, where λ_i are the eigenvalues of A.
Proof. The proof is very similar to the Theorem 4.1, so is omitted here.
Corollary 4.2. If the local controllers u_i are constructed as

$$u_i = -ck\Gamma e_i(t),$$

$k > \varpi\lambda_i + \frac{1}{c}$ and $0 < \delta < 1$, then the delayed NMAS(4.1) will achieve bounded consensus and the consensus set is depicted as follows

$$M = \{e(t)|\|e(t)\| \leq \frac{\sum_{j=1}^{N} \alpha_j \overline{lim}_{t\to\infty} \|w_{ji}(t)\|}{1-\delta}\}, \tag{4.22}$$

namely, $e_i(t) = x_i(t) - s(t) \to \Omega$ as $t \to \infty$, where λ_i are the eigenvalues of A and ϖ is defined as (4.19).

4.5 Exact consensus via nonlinear feedback

The designed controllers in the Theorem 4.2 make the NMAS (4.1) achieve bounded consensus instead of exact consensus, the following theorem will consider how to design controllers to guarantee the NMAS achieve exact consensus.

Theorem 4.3. The NMAS(4.1) will achieve exact consensus if the local controllers u_i are constructed as

$$u_i = -ck\Gamma e_i(t) - \rho sgn(e_i(t)) \tag{4.23}$$

for $i = 1, 2, \cdots, N$, where $sgn(e_i(t)) = [sgn(e_{i1}(t)), sgn(e_{i2}(t)), \cdots, sgn(e_{in}(t))]^T$, with $sgn(.)$ are signum function, and furthermore, the controller gains are designed to satisfy the bounds

$$k > \mathtt{max}\{\lambda_i + \frac{1}{c}, 0\}$$
$$\rho > \frac{\sum_{j=1}^{N} \alpha_j \|w_{ji(t)}\|}{\sum_{j=1}^{N} \|w_{ji(t)}\|}$$

for any i.

Proof. Rewriting the error system (4.11) in vector form one gets:

$$\dot{\bar{e}}(t) = \bar{B}(t) + c\Gamma\bar{e}(t)(A^T - K) - \rho sgn(\bar{e}_i(t)), \tag{4.24}$$

where $\bar{e}(t) = [e_1(t), e_2(t), \cdots, e_N(t)] \in R^{n\times N}$, $\bar{B}(t) = [\hat{B}_1(t), \hat{B}_2(t), \cdots, \hat{B}_N(t)] \in R^{n\times N}$, $\hat{B}_i(t) = B_i x_i(t) - \frac{1}{N}\sum_{k=1}^{N} B_k x_k(t)$, $i = 1, 2, \cdots, N$, $K = diag\{k, k, \cdots, k\} \in R^{N\times N}$ and $sgn(\bar{e}(t)) = [sgn(e_1(t)), sgn(e_2(t)), \cdots, sgn(e_N(t))] \in R^{n\times N}$.

Given that the connectivity matrix satisfies Lemma 4.1, there are two matrices,

$\Omega = (w_1, w_2, \cdots, w_N) \in R^{N\times N}$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_N) \in R^{N\times N}$ such that:

$$A = \Omega^T \Lambda \Omega, \tag{4.25}$$

where λ_i and w_i are the $i-$th eigenvalue and associated eigenvector of A respectively. With $\Omega^T\Omega = I_N$, the $N-$dimensional identity matrix.
Using a change of variables $\bar{\eta}(t) = \bar{e}(t)\Omega^T$, the error dynamics become

$$\dot{\bar{\eta}}(t) = [\bar{B}(t) - \rho sgn(\bar{\eta}(t)\Omega)]\Omega^T + c\Gamma\bar{\eta}(t)(\Lambda - K),$$

where $\bar{\eta}(t) = (\eta_1(t), \eta_2(t), \cdots, \eta_N(t))$, with $\eta_i(t) = \bar{e}(t)w_i^* \in R^n$ and $w_i^* = [w_{1i}, w_{2i}, \cdots, w_{Ni}]^T \in R^{N\times 1}$, or equivalently,

$$\dot{\bar{\eta}}_i(t) = (\bar{B}(t) - \rho sgn(\bar{\eta}(t)\Omega))w_i^* + c\Gamma\eta_i(t)(\lambda_i - k), \quad i = 1, 2, \cdots, N. \tag{4.26}$$

The stability of the error dynamics (4.24) around the zero fixed point can be determine using the Lyapunov candidate function:

$$V = \frac{1}{2}\sum_{i=1}^{N} \eta_i^T(t)\eta_i(t). \tag{4.27}$$

The time derivative of V along the trajectories of the error dynamics in (4.16) is given by

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N}(\eta_i^T(t)\bar{B}(t)w_i^*(t) - \bar{\eta}_i^T(t)\rho sgn(\bar{\eta}(t)\Omega)w_i^*(t) + \eta_i^T(t)c(\lambda_i - k)\Gamma\eta_i(t)) \\
&\leq \sum_{i=1}^{N}(\|\eta_i(t)\|^T\|\bar{B}(t)w_i^*(t)\| - \rho\|\bar{\eta}_i(t)\|^T\|sgn(\bar{\eta}(t)\Omega)w_i^*(t)\| \\
&\quad + c(\lambda_i - k)\|\eta_i(t)\|^T\Gamma\|\eta_i(t)\|).
\end{aligned}$$

Considering the bounds of each term of $\dot{V}$. From (4.12) one has the bound of the first term:

$$\|\bar{B}(t)w_i^*(t)\| \leq \|\sum_{j=1}^{N} B_j(t)w_{ji}(t)\| \leq \|\sum_{j=1}^{N} \alpha_j w_{ji}(t)\|.$$

The bound for the second term is given by:

$$\|sgn(\bar{\eta}(t)\Omega)w_i^*(t)\| \leq \sum_{j=1}^{N} \|sgn(\bar{\eta}(t)w_j(t))\|\|w_{ji}(t)\| \leq \sum_{j=1}^{N} \|w_{ji}(t)\|.$$

The third term is quadratic and will be negative if the coefficient is negative ($c(\lambda_i - k) < 0$) for any i. The bound on the third term can be expressed as

$$c(\lambda_i - k)\|\eta_i(t)\|^T \Gamma \|\eta_i(t)\| \leq -\|\eta_i(t)\|^T \|\eta_i(t)\|.$$

From the above results the time derivative V is bounded by

$$\dot{V} \leq \sum_{i=1}^{N} (\|\eta_i(t)\|^T (\sum_{j=1}^{N} \alpha_j \|w_{ji}(t)\| - \rho \sum_{j=1}^{N} \|w_{ji}(t)\|) - \|\eta_i(t)\|^T \|\eta_i(t)\|).$$

For V to be negative, the discontinuous gain must satisfy $\rho > \frac{\sum_{j=1}^{N} \alpha_j \|w_{ji}(t)\|}{\|w_{ji}(t)\|}$ for any i. Then the error dynamics (4.26) are globally uniformly asymptotically stable about the zero fixed point ($\bar{\eta}(t) = 0$), which implies that the NMAS (4.1) under the controllers (4.23) achieve consensus.

Corollary 4.3. The delayed NMAS(4.1) will achieve exact consensus if the local controllers u_i are constructed as

$$u_i = -ck\Gamma e_i(t) - \rho sgn(e_i(t)) \tag{4.28}$$

for $i = 1, 2, \cdots, N$, where $sgn(e_i(t)) = [sgn(e_{i1}(t)), sgn(e_{i2}(t)), \cdots, sgn(e_{in}(t))]^T$, with $sgn(.)$ are signum function, and furthermore, the controller gains are designed to satisfy the bounds

$$k > \mathtt{max}\{\varpi\lambda_i + \frac{1}{c}, 0\}$$
$$\rho > \frac{\sum_{j=1}^{N} \alpha_j \|w_{ji(t)}\|}{\sum_{j=1}^{N} \|w_{ji(t)}\|}$$

for any i, where ϖ is defined as (4.19).

Remark 4.1. The above bounded consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical nodes to the cases of non-identical nodes. Because of the complexity of the consensus problems for non-identical nodes, we only obtain here sufficient conditions instead of sufficient and necessary condition. However, the conditions obtained here are easy to verify and we can easily construct appropriate numerical simulation example to verify the effectiveness of the proposed results. Comparing the above three theorems, it can be seen that the linear negative feedback can't guarantee the NMAS achieve exact consensus, at the same time, the boundary of the convergence set can be evaluated respectively.

4.6 An Example

In this section, we will construct an example to demonstrate the proposed results above. The problem is to guarantee 11 agents to follow desired curves in a 2-dimensional system of coordinate.

The agent dynamics can be chosen as follows

$$\dot{x}_i(t) = B_i x_i(t), \quad i = 1, 2, \cdots, 11, \tag{4.29}$$

where

$$B_i = \begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix}, \quad i = 1, 2, \cdots, 11.$$

The communication coupling matrix A and the inner coupling matrix are

$$A = \begin{pmatrix} -8 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -8 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -6 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & -7 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -5 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -10 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -6 \end{pmatrix}, \Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively.

Given the initial values of 11 agents as $(10\ 5\ \ -10)^T$, $(12\ 6\ \ -12)^T$, $(14\ 7\ \ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$ and $(-10\ 15\ 10)^T$ respectively. We may verify the conditions of Theorem 4.3 and Corollary 4.3 readily, then the consensus of the NMAS is achieved. Simulation results are depicted in Fig 4.1 to Fig 4.6 for $c = 1$, $k = 0.5$ and $\tau = 0.055$. The former of each figure indicates the NMAS without coupling time-delay, and the latter indicates the delayed NMAS.

The simulation curves in Fig 4.1-Fig 4.6 show that the dynamics of the NMAS in different time scale with and without control respectively. The average state trajectory

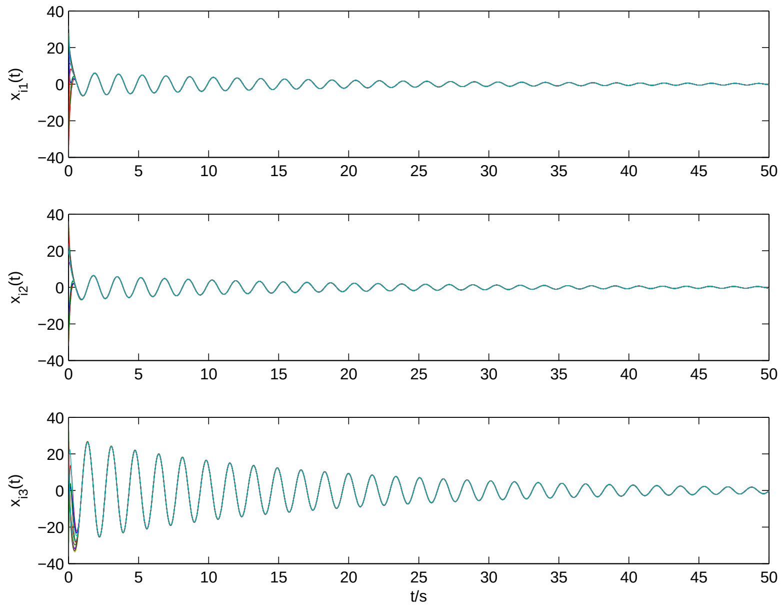

Fig 4.1.1 The dynamics of all agents without delay and control.

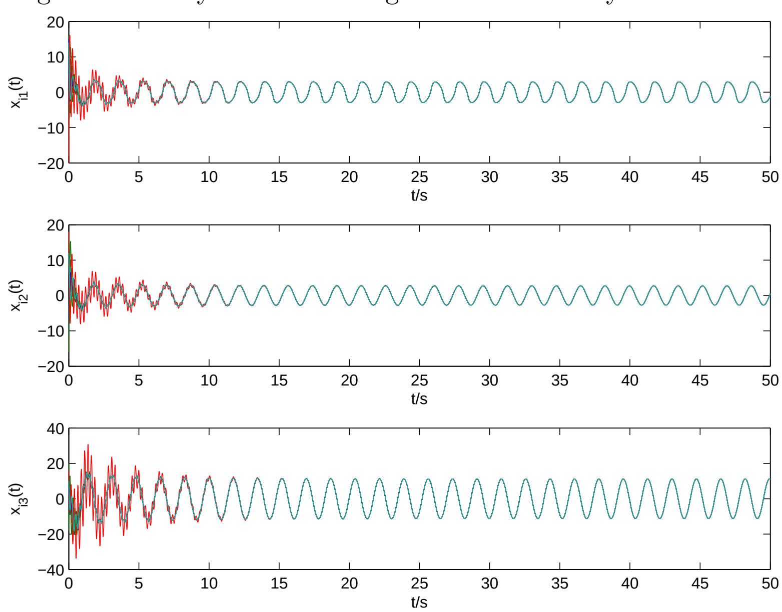

Fig 4.1.2 The dynamics of all agents with delay but without control.

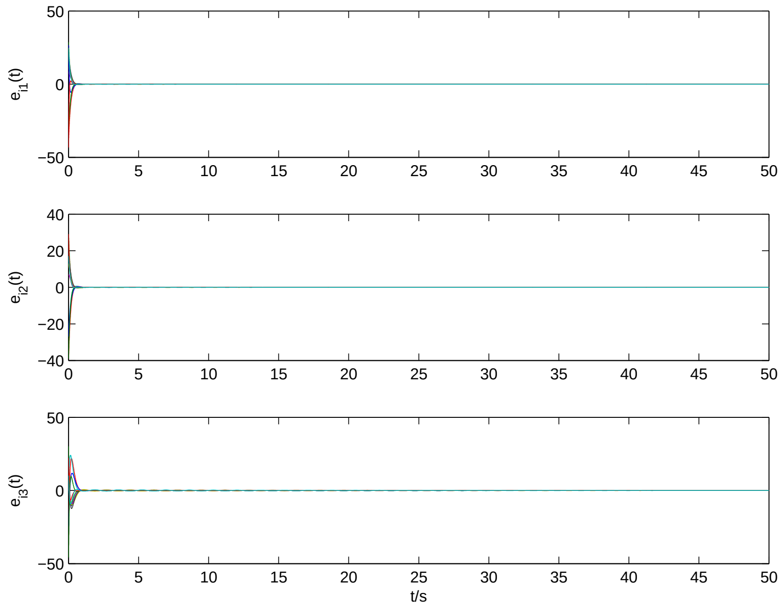

Fig 4.2.1 The consensus error dynamics of each agent without delay and control.

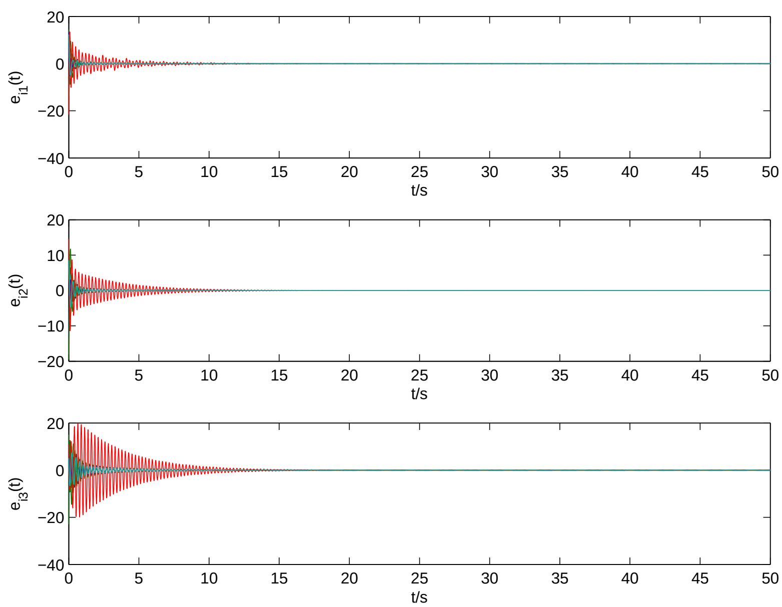

Fig 4.2.2 The consensus error dynamics of each agent with delay but without control.

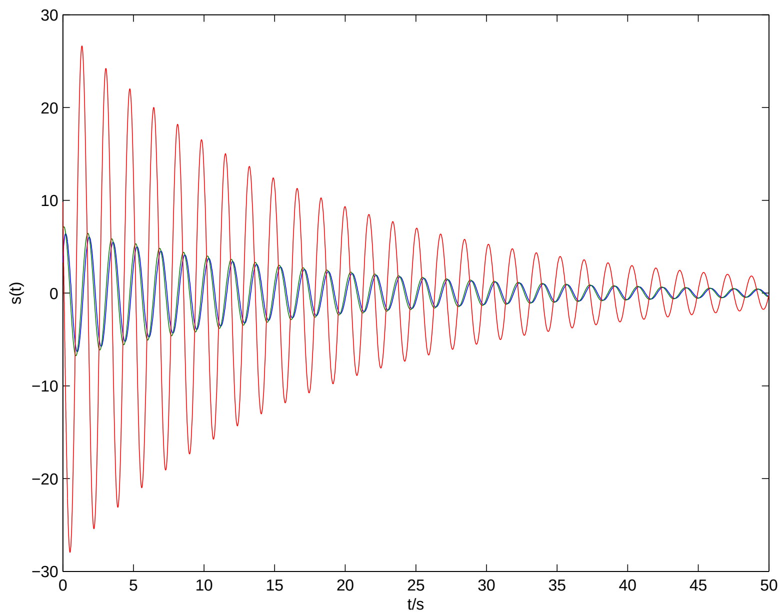

Fig 4.3.1 The average state trajectory $s(t)$ without delay control.

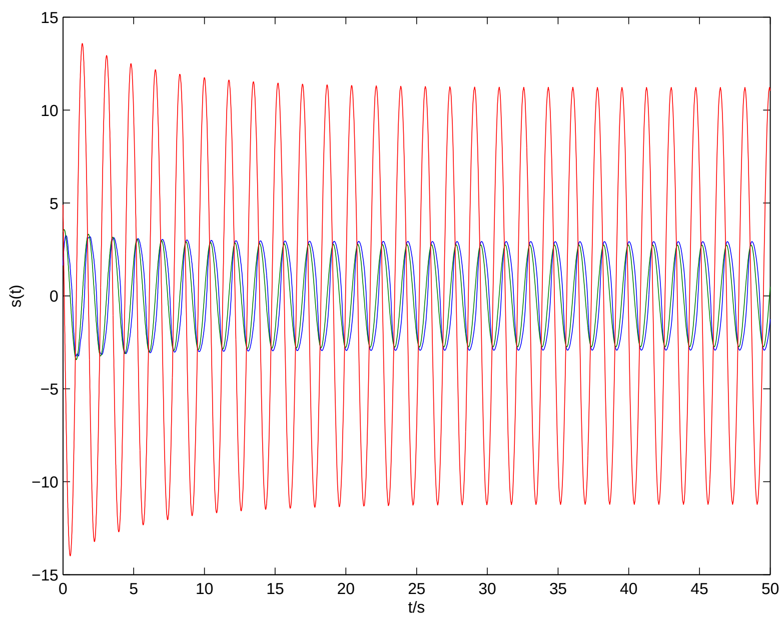

Fig 4.3.2 The average state trajectory $s(t)$ with delay but without control.

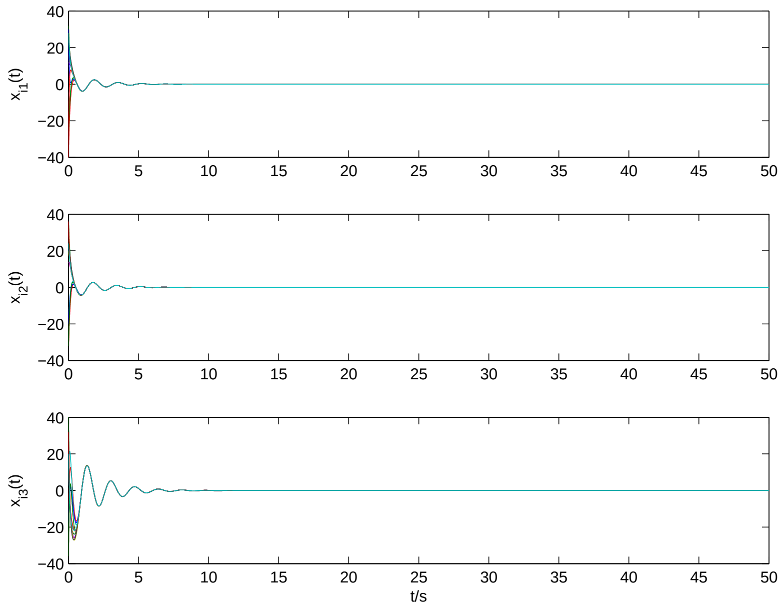

Fig 4.4.1 The dynamics of all agents without delay but with control.

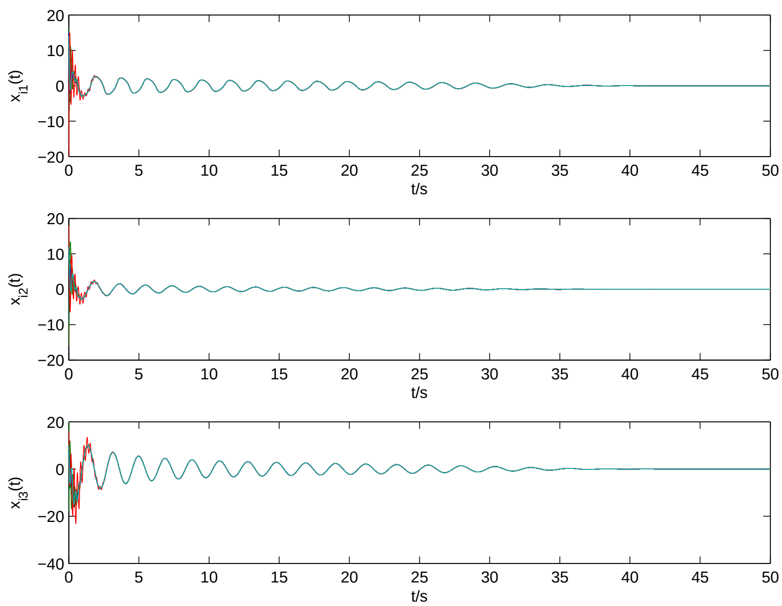

Fig 4.4.2 The dynamics of all agents with delay and control.

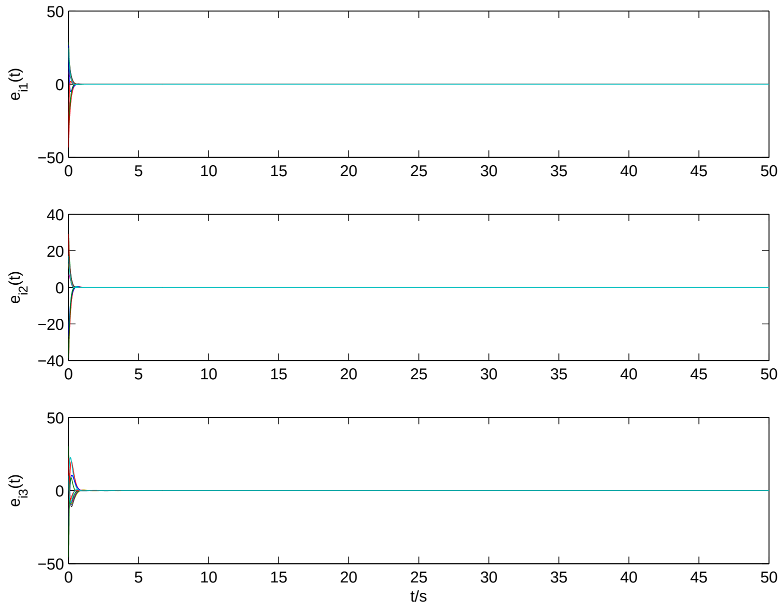

Fig 4.5.1 The consensus error dynamics of each agent without delay but with control.

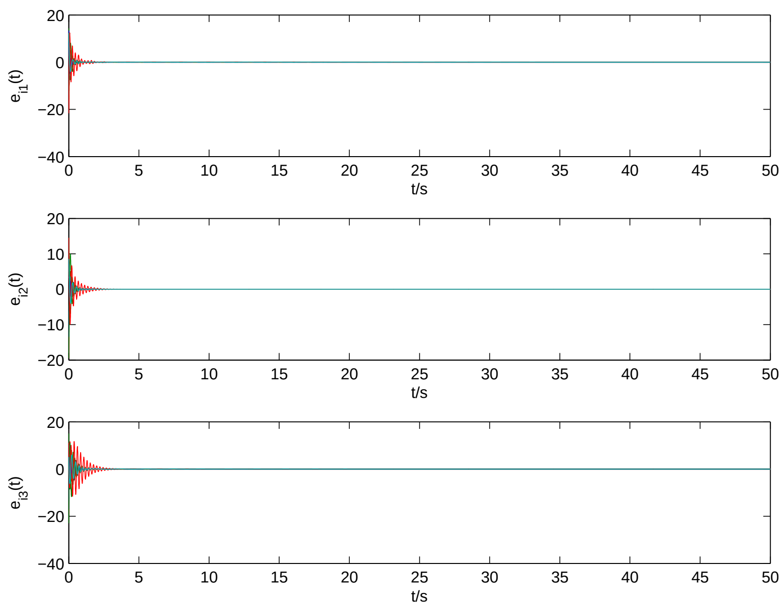

Fig 4.5.2 The consensus error dynamics of each agent with delay and control.

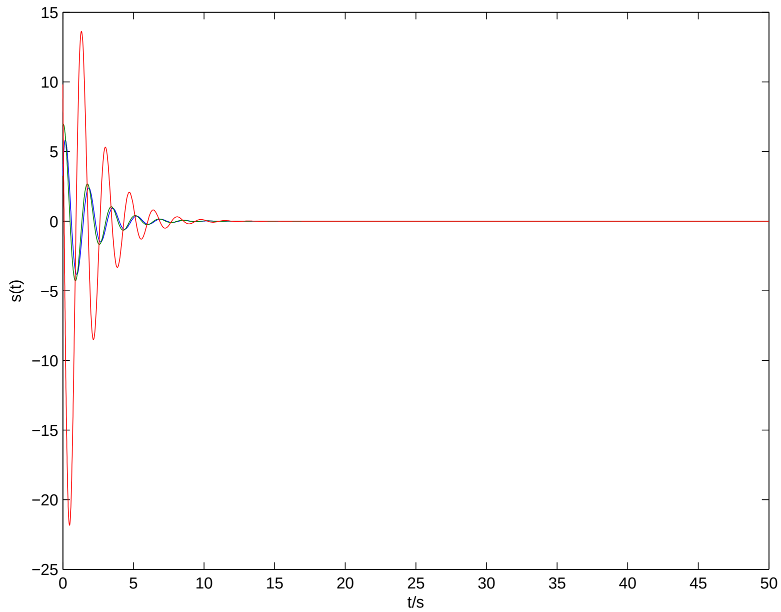

Fig 4.6.1 The average state trajectory $s(t)$ without delay but with control.

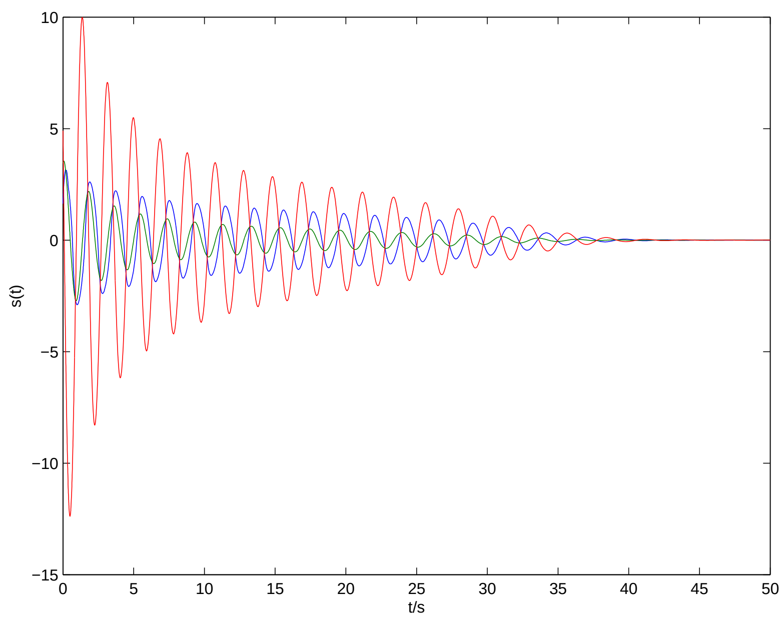

Fig 4.6.2 The average state trajectory $s(t)$ with delay and control.

$s(t)$ is chosen as the desired moving trajectory and is depicted in Fig 4.3 and Fig 4.6 respectively. These simulation curves show that all agents eventually move with the desired state trajectory in the sense of boundedness on the condition that there is not control in the NMAS or by using linear negative feedback. The exact consensus can be guaranteed by means of nonlinear controllers.

4.7 Conclusions

In this chapter, we've investigated the consensus problems of NMAS with different agent dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on Lyapunov stability theorem. The methods presented here have several distinct features. Firstly, they are very simple in form, but are more effective to resolve the consensus problem with non-identical agent dynamics. Secondly, the proposed nonlinear controllers can guarantee the NMAS achieve exact consensus instead of in terms of boundedness. It should be noted that the conditions are still restrictive here, further work will focus on these problems. In addition, an obvious limitation of the proposed method is the fact the requires the same number of controllers than agents, in a work to be reported elsewhere this controlled consensus design is combined with a pinning control strategy, providing a reduction on the number of agents where controlled action in taken. Yet another aspect of interest to be considered as future work is determining conditions for the existence of an appropriate coordinate transformation to translate the nonlinear agent dynamics to linear ones such that this result is applicable for a more general class of oscillators.

Chapter 5

Global Bounded Consensus of NMAS with Different Agents and Communication Time-Delay Topology

This chapter will investigate the global bounded consensus problem of NMAS exhibiting nonlinear, non-identical agent dynamics and communication time-delay. Globally bounded consensus conditions for both delay-independent and delay-dependent conditions based on the Lypunov-Krasovskii functional method are derived. In addition, globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived. The proposed consensus criteria ensures that all agents eventually move along the desired trajectories in the sense of boundedness. The proposed consensus criteria generalizes the case of identical agent dynamics to the case of non-identical agent dynamics and many related results in this area can be viewed as special cases of the present results. Finally, the effectiveness of the theoretical results will be demonstrated by means of numerical simulations.

5.1 Introduction

The agent dynamics in most existing works are often restricted to linear and identical ones. However, this is not always the case in practice. The consensus problem of NMAS with non-identical agent dynamics and time-delay topology is much more complicated than the identical case and few results have been reported to date.

The similarity between the consensus of NMAS and the synchronization of complex dynamical networks shows us a way forward. Therefore, if we use the ideas in the synchronization problem of complex dynamical networks properly, then consensus problems of NMAS are solvable.

This chapter will focus on the global consensus problems of NMAS with non-identical agent dynamics and time-delay topology. The behavior of the NMAS with non-identical agent dynamics is much more complicated than the identical case. Usually, no common equilibrium for all agents exists even if each agent has an equilibrium, neither does a consensus manifold exist in the classical sense. Consensus of a NMAS with identical agents is usually described in terms of (asymptotically) identical dynamical evolution of state variables of every agent in the NMAS, which is easy to understand. However, this collective behavior, called exact consensus no longer exists in the NMAS with non-identical agents due to the difference between the dynamics of the agents. Furthermore, we can't decompose the NMAS with non-identical agent dynamics into a number of lower dimensional systems exactly like the identical-agent case. Yet, a NMAS with non-identical agents may still exhibit some kinds of consensus behaviors which are far from being fully understood, and very few results have been reported by now. Certain reasonable and satisfactory boundedness of state motions errors between different agents can be taken as useful consensus properties.

The rest of this chapter is organized as follows. A continuous-time NMAS model with non-identical agent dynamics and communication time-delay topology is presented in section 5.2. The main results including delay-independent and delay-dependent bounded consensus criterion are derived in section 5.3. In section 5.4, globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived. Numerical simulation examples are given to verify the effectiveness of the proposed results in section 5.5, followed by conclusions in section 5.6.

5.2 Problem Description

Consider a NMAS consisting of N non-identical agents with communication delay:

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma x_j(t-\tau), \quad i = 1, 2, \cdots, N, \tag{5.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i(t)) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix, and where $\gamma_{ij} \neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N\times N}$(which is symmetric and irreducible) represents the communication topology relation of the NMAS, and is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j\neq i} a_{ij}$. τ is a constant coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field

$$\bar{f}(x(t)) = \frac{1}{N}\sum_{k=1}^{N} f_k(x(t)). \tag{5.2}$$

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N} \sum_{k=1}^{N} x_k(t). \tag{5.3}$$

We now discuss the problem of global consensus for the NMAS (5.1). The consensus problem formulation in the present paper is quite different from many others, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. The consensus problem here will be depicted instead via certain boundedness of $x_i(t) - x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. This better reflects reality since it is impossible for NMAS (5.1) to achieve exact consensus. To address this case we will focus on making the states of all agents converge to a bounded set.

Lemma 5.1 ([94, 95, 96, 97, 98, 99, 100]): Assume that $a(.) \in R^{n_a}$, $b(.) \in R^{n_b}$ and $M(.) \in R^{n_a \times n_b}$ are defined on an interval Ω. Then, for any matrices $X \in R^{n_a \times n_a}$, $Y \in R^{n_a \times n_b}$ and $Z \in R^{n_b \times n_b}$, the following inequality holds:

$$-2 \int_{\Omega} a^T(\alpha) M b(\alpha) d\alpha \leq \int_{\Omega} \begin{pmatrix} a(\alpha) \\ b(\alpha) \end{pmatrix}^T \begin{pmatrix} X & Y - M \\ * & Z \end{pmatrix} \begin{pmatrix} a(\alpha) \\ b(\alpha) \end{pmatrix} d\alpha, \tag{5.4}$$

where

$$\begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} \geq 0.$$

We now introduce some notations and definitions.

Let "$\otimes$" be Kronecker product.

5.3 Global Bounded Consensus Analysis

Define the error vector

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \cdots, N. \tag{5.5}$$

Obviously, $\sum_{i=i}^{N} e_i = 0$ and $\frac{c}{N} \sum_{k=1}^{N} \sum_{j \in \mathcal{N}_i}^{N} a_{kj} \Gamma x_j(t-\tau) = 0$, then the NMAS (5.1) can be rewritten in terms of e_i as

$$\dot{e}_i(t) = f_i(s(t) + e_i(t)) - \frac{1}{N} \sum_{k=1}^{N} f_k(s(t) + e_k(t)) + c \sum_{j \in \mathcal{N}_i}^{N} a_{ij} \Gamma e_j(t-\tau). \tag{5.6}$$

The following work will focus on simplifying the error NMAS (5.6) by means of a series of transformations using a procedure similar to [88].

Applying the Newton-Leibniz formula, error NMAS (5.6) can be further written as

$$\begin{aligned}\dot{e}_i(t) = {} & D\bar{f}(s(t))e_i(t) + c \sum_{j \in \mathcal{N}_i}^{N} a_{ij} \Gamma e_j(t-\tau) \\ & + \int_0^1 (Df_i(s(t) + \tau e_i(t)) - D\bar{f}(s(t)))e_i(t) d\tau \\ & - \frac{1}{N} \sum_{k=1}^{N} \int_0^1 Df_k(s(t) + \tau e_k(t))e_k(t) d\tau + f_i(s(t)) - \bar{f}(s(t)).\end{aligned} \tag{5.7}$$

If we consider the linearized NMAS of (5.1), we have

$$\begin{aligned}\dot{e}_i(t) = {} & D\bar{f}(s(t))e_i(t) + c \sum_{j \in \mathcal{N}_i}^{N} a_{ij} \Gamma e_j(t-\tau) + (Df_i(s(t)) - D\bar{f}(s(t)))e_i(t) \\ & - \frac{1}{N} \sum_{k=1}^{N} Df_k(s(t))e_k(t) + f_i(s(t)) - \bar{f}(s(t)).\end{aligned} \tag{5.8}$$

Let $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$, then (5.7) becomes

$$\dot{e}(t) = I_N \otimes D\bar{f}(s)e(t) + cA \otimes \Gamma e(t-\tau) + I(t)e(t) - \frac{1}{N}H(t)e(t) + F(t), \quad (5.9)$$

where

$$\begin{aligned} I(t) = diag\{ &\int_0^1 (Df_1(s(t)+\tau e_1(t)) - D\bar{f}(s(t)))d\tau \cdots \\ &\int_0^1 (Df_N(s(t)+\tau e_N(t)) - D\bar{f}(s(t)))d\tau\}, \end{aligned}$$

$$H(t) = \begin{pmatrix} \int_0^1 Df_1(s(t)+\tau e_1(t))d\tau & \cdots & \int_0^1 Df_N(s(t)+\tau e_N(t))d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s(t)+\tau e_1(t))d\tau & \cdots & \int_0^1 Df_N(s(t)+\tau e_N(t))d\tau \end{pmatrix},$$

$$F(t) = \begin{pmatrix} f_1(s(t)) - \bar{f}(s(t)) \\ \vdots \\ f_N(s(t)) - \bar{f}(s(t)) \end{pmatrix}.$$

Since A is symmetric and irreducible, according to Lemma 4.1, there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \cdots, \Phi_N)$, such that (**??**) is satisfied. This together with $\omega(t) = (\Phi^T \otimes I_n)e(t)$ gives

$$\begin{aligned} \dot{\omega}(t) = &(\Phi^T \otimes I_n)(I_N \otimes D\bar{f}(s))(\Phi \otimes I_n)\omega(t) \\ &+ (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)\omega(t-\tau) + (\Phi^T \otimes I_n)F(t) \\ &+ (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) - \frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)\omega(t). \end{aligned}$$

Note that

$$H(t) = \sqrt{N}\sum_{k=1}^{N}\left[(\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau\right], \tag{5.10}$$

where $\bar{\Phi}_k$ stands for the matrix with its k-th column equals Φ_1 and the rest of its elements are zero, then we have

$$\begin{aligned}&\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)\\&\qquad = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau(\Phi \otimes I_n),\end{aligned} \tag{5.11}$$

where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the rest of its elements are zero.

Thus, a simple calculation gives

$$\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\begin{pmatrix}\Upsilon_k\\ \mathsf{0}\end{pmatrix} \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau, \tag{5.12}$$

where $\Upsilon_k \in R^{1\times N}$ and $\mathsf{0} \in R^{(N-1)\times N}$.

Therefore,

$$\begin{aligned}\dot{\omega}(t) &= I_N \otimes D\bar{f}(s)\omega(t) + c\Lambda \otimes \Gamma\omega(t-\tau) + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)\\&\quad - \begin{pmatrix}*\\ \mathsf{0}\end{pmatrix}\omega(t) + (\Phi^T \otimes I_n)F(t).\end{aligned} \tag{5.13}$$

Since $\omega_1 \equiv 0$, we only need to consider $\omega_2(t), \omega_3(t), \cdots, \omega_N(t)$. Rewriting (5.13) in the component form we have

$$\dot{\omega}_i(t) = D\bar{f}(s(t)\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)$$
$$+ (\Phi_i^T \otimes I_n)F(t),\ i = 2, 3, \cdots, N. \tag{5.14}$$

So far, we have transferred the consensus problem of NMAS (5.1) to the stability problem of the $N-1$ of $n-$dimensional systems.

In the following, a time-independent global consensus criteria is derived for the NMAS (5.1).

Theorem 5.1 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$a\|x(t)\|^2 \quad \leq x^T(t)P_i(t)x(t) + \int_{t-\tau}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \leq b\|x(t)\|^2,$$
$$\forall t \in R^+,\ x \in R^n, i = 2, 3, \cdots, N, \tag{5.15}$$

$$\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + Q_i$$
$$+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \zeta I \leq 0,\ i = 1, 2, \cdots, N, \tag{5.16}$$

$$\|I(t)\| \leq \gamma,\ i = 1, 2, \cdots, N. \tag{5.17}$$

Then the system (5.45) converges to the set

$$M = \{e(t) | \|e(t)\| \leq \frac{2b}{a}\frac{\beta\overline{lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\} \tag{5.18}$$

for any fixed time delay $\tau > 0$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, $\beta = (\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}}$, $\zeta > 2\gamma\beta$ and $\mu(t) = \|F(t)\|$ is bounded. Furthermore, the

NMAS (5.1) achieves bounded consensus for any fixed time delay $\tau > 0$.

Proof. Choose the following Lyapunov-Krasovskii functional as

$$V(w_i(t), t) = \sum_{i=2}^{N} V_i(w_i(t), t), \tag{5.19}$$

$$V_i(w_i(t), t) = w_i^T(t)P_i(t)w_i(t) + \int_{t-\tau}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha, \quad i = 2, 3, \cdots, N. \tag{5.20}$$

Differentiating (5.20) along the trajectory of (5.47) gives

$$\begin{aligned}\dot{V}_i(w_i(t), t) &= w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + Q_i)w_i(t) \\ &+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \\ &+ 2w_i^T(t)(c\lambda_i P_i(t)\Gamma)w_i(t-\tau) - w_i^T(t-\tau)Q_i w_i(t-\tau).\end{aligned} \tag{5.21}$$

Applying the Young Inequality to the equality (5.21), results in

$$\begin{aligned}\dot{V}_i(w_i(t), t) \leq\ & w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) \\ &+ Q_i + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i(t) \\ &+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w(t) \\ &+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t).\end{aligned} \tag{5.22}$$

Condition (5.16) implies that the first term on the right hand side of (5.22) satisfies

$$\begin{aligned}&w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + Q_i \\ &\quad + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i(t) \leq -\zeta\|w_i(t)\|^2.\end{aligned} \tag{5.23}$$

Applying condition (5.17) we know the second term on the right hand side of (5.22) satisfies

$$2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \leq 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|. \tag{5.24}$$

The third term on the right hand side of (5.22) satisfies

$$2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i(t)\|. \tag{5.25}$$

Since $V(w(t),t) = \sum_{i=2}^{N} V_i(w_i(t),t)$, we have

$$\begin{aligned}
\dot{V}(w(t),t) &= \sum_{i=2}^{N} \dot{V}_i(w_i(t),t) \\
&\leq \sum_{i=2}^{N}(-\zeta\|w_i(t)\|^2) + 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\| + 2\mu(t)\|P_i(t)\|\|w_i(t)\| \\
&= -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\| + \mu(t))\sum_{i=2}^{N}\|w_i(t)\|\|P_i(t)\| \\
&\leq -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\| + \mu(t))\|w(t)\|(\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}} \\
&= \|w(t)\|((2\gamma\beta - \zeta)\|w(t)\| + 2\beta\mu(t)).
\end{aligned} \tag{5.26}$$

Thus, when

$$\|w(t)\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \tag{5.27}$$

we have

$$\dot{V}(w(t),t) \leq -\delta\|w(t)\|^2. \tag{5.28}$$

Applying Lemma 4.2 completes the proof.

Corollary 5.1 We have an asymptotic consensus criterion in the classical sense when $\overline{lim}_{t\to\infty}\mu(t) = 0$. In particular, we have $\mu(t) \equiv 0$ when all agents have the same dynamics, i.e., $f_i(x_i(t)) = f(x(t))$. In such a case, applying theorem 5.1 to the linearized network, which is equivalent to taking $\gamma = 0$ in (5.17), immediately achieves the universal consensus criteria existing in many literatures. Therefore, Theorem 5.1 covers the existing criteria of networks with identical agent dynamics as a special case.

Remark 5.1. The above result is a delay-independent globally consensus criterion and the ultimate convergence bound is evaluated by means of (5.18). Theorem 5.1 guarantees that all agents move along the desired trajectory described by $s(t)$ in terms of certain boundedness, i.e., the consensus achieved here is just approximate instead of exact, in fact, to achieve exact consensus is impossible for such a case.

Next, we will provide delay-dependent criterion for the proposed problem.

Theorem 5.2 Suppose that (5.15) and (5.17) in Theorem 5.1 are satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Pi_i > 0$, $\Pi_i > 0$, X_i, Y_i and Z_i of appropriate dimensions such that

$$\Xi = \begin{pmatrix} \Xi_{11} & c\lambda_i P_i(t)\Gamma - Y_i + hc\lambda_i D\bar{f}(s(t))^T Z_i\Gamma \\ * & \Pi_i^{-1} + \Sigma_i^{-1} - Q_i + hc^2\lambda_i^2\Gamma^T Z_i\Gamma \end{pmatrix} < 0, \tag{5.29}$$

where $\Xi_{11} = \dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + hX_i + Y_i^T + Y_i + Q_i + hD\bar{f}(s(t))^T Z_i D\bar{f}(s(t))$ and

$$\begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{5.30}$$

for $i = 2, 3, \cdots, N$, then the NMAS (5.1) will achieve bounded consensus for the time-invariant delay $\tau \in [0, h]$ for some $h < \infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V(w_i(t), t) = \sum_{i=2}^{N} \sum_{k=1}^{3} V_k(w_i(t), t), \tag{5.31}$$

where

$$\begin{aligned}
V_1(w_i(t), t) &= w_i^T(t) P_i(t) w_i(t), \\
V_2(w_i(t), t) &= \int_{-\tau}^{0} \int_{t+\beta}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha d\beta, \\
V_3(w_i(t), t) &= \int_{t-\tau}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha.
\end{aligned}$$

The i-th $(i = 2, 3, \cdots, N)$ equation in system (5.47) can be written as

$$\begin{aligned}
\dot{\omega}_i(t) = (D\bar{f}(s(t)) + c\lambda_i\Gamma)\omega_i(t) - c\lambda_i\Gamma \int_{t-\tau}^{t} \dot{w}_i(\alpha) d\alpha \\
+ (\Phi_i^T \otimes I_n) I(t) (\Phi \otimes I_n) \omega(t) + (\Phi_i^T \otimes I_n) F(t), \quad i = 2, 3, \cdots, N, \quad (5.32)
\end{aligned}$$

and, thus, the derivative of $V_1(w_i(t), t)$ satisfies

$$\begin{aligned}
\dot{V}_1(w_i(t), t) = w_i^T(t)(\dot{P}_i(t) + P_i(t)(D\bar{f}(s(t)) + c\lambda_i\Gamma) \\
+ (D\bar{f}(s(t)) + c\lambda_i\Gamma)^T P_i(t)) w_i(t) \\
- 2c\lambda_i w_i^T(t) P_i(t) \Gamma \int_{t-\tau}^{t} \dot{w}_i(\alpha) d\alpha \\
+ 2w_i^T(t) P_i(t) (\Phi_i^T \otimes I_n) I(t) (\Phi_i \otimes I_n) w(t) \\
+ 2w_i^T(t) P_i(t) (\Phi_i^T \otimes I_n) F(t). \quad (5.33)
\end{aligned}$$

Defining $a(.)$, $b(.)$ and M in (5.4) as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$ and $M =$

$c\lambda_i P_i(t)\Gamma$ for all $\alpha \in [t-\tau, t]$ and then applying Lemma 5.1 results in

$$\begin{aligned}
\dot{V}_1(w_i(t),t) \leq\ & w_i^T(t)[\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + hX_i \\
& + Y_i^T + Y_i]w_i(t) + 2w_i^T(t)(c\lambda_i P_i(t)\Gamma - Y_i)w_i(t-\tau) + \int_{t-\tau}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha \\
& + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t).
\end{aligned} \tag{5.34}$$

Moreover, since

$$\begin{aligned}
\dot{V}_2(w_i(t),t) =\ & \tau[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\
& + (\Phi_i^T \otimes I_n)F(t)]^T Z_i[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau) \\
& + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) + (\Phi_i^T \otimes I_n)F(t)] - \int_{t-\tau}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha,
\end{aligned} \tag{5.35}$$

the above equality can be enlarged as

$$\begin{aligned}
\dot{V}_2(w_i(t),t) \leq\ & h[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)]^T Z_i[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)] \\
& + 2h(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\
& + 2h(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
& + 2h(c\lambda_i\Gamma\omega_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\
& + 2h(c\lambda_i\Gamma\omega_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
& + 2h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
& + h((\Phi_i^T \otimes I_n)F(t))^T Z_i((\Phi_i^T \otimes I_n)F(t)) \\
& + h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)) \\
& - \int_{t-\tau}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha.
\end{aligned} \tag{5.36}$$

$$\dot{V}_3(w_i(t),t) = w_i^T(t)Q_iw_i(t) - w_i^T(t-\tau)Q_iw_i(t-\tau). \tag{5.37}$$

The derivative of $V(w_i(t),t)$ is

$$\begin{aligned}
&\sum_{k=1}^{3}\dot{V}_k(w_i(t),t)\\
&\leq w_i^T(t)[\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^TP_i(t) + hX_i + Y_i^T + Y_i]w_i(t)\\
&+ 2w_i^T(t)(c\lambda_iP_i(t)\Gamma - Y_i)w_i(t-\tau) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t)\\
&+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) + 2h(D\bar{f}(s(t))\omega_i(t))^TZ_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)\\
&+ h[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)]^TZ_i[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)]\\
&+ 2h(D\bar{f}(s(t))\omega_i(t))^TZ_i(\Phi_i^T \otimes I_n)F(t)\\
&+ 2h(c\lambda_i\Gamma\omega_i(t-\tau))^TZ_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)\\
&+ 2h(c\lambda_i\Gamma\omega_i(t-\tau))^TZ_i(\Phi_i^T \otimes I_n)F(t)\\
&+ 2h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^TZ_i(\Phi_i^T \otimes I_n)F(t)\\
&+ h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^TZ_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))\\
&+ h((\Phi_i^T \otimes I_n)F(t))^TZ((\Phi_i^T \otimes I_n)F(t)) + w_i^T(t)Q_iw_i(t) - w_i^T(t-\tau)Q_iw_i(t-\tau).
\end{aligned} \tag{5.38}$$

Applying the Young Inequality, then we have

$$\begin{aligned}
&2h(c\lambda_i\Gamma\omega_i(t-\tau))^TZ_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \leq w_i^T(t-\tau)\Pi_i^{-1}w_i(t-\tau)\\
&+ h^2c^2\lambda_i^2w^T(t)((\Phi \otimes I_n)^TI(t)(\Phi_i^T \otimes I_n)^TZ_i\Gamma\Pi_i\Gamma^TZ_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n))w(t),
\end{aligned} \tag{5.39}$$

and $h(c\lambda_i\Gamma\omega_i(t-\tau))^TZ_i(\Phi_i^T \otimes I_n)F(t) \leq w_i^T(t-\tau)\Sigma_i^{-1}w_i(t-\tau) + h^2c^2\lambda_i^2F^T(t)(\Phi_i^T \otimes I_n)^TZ_i\Gamma\Sigma_i\Gamma^TZ_i(\Phi_i^T \otimes I_n)F(t)$.

Applying (5.15) and (5.17) to the above inequality results in:

$$
\begin{aligned}
\dot{V}(w_i(t),t) \leq & \sum_{i=2}^{N} \begin{pmatrix} w_i(t) \\ w_i(t-\tau) \end{pmatrix}^T \Xi \begin{pmatrix} w_i(t) \\ w_i(t-\tau) \end{pmatrix} \\
& + \|w\|((2\gamma\beta + 2hr\|D\bar{f}(s(t))\| \sum_{i=2}^{N} \lambda_{max}(Z_i) + 2h\mu(t)\|D\bar{f}(s(t))\| \sum_{i=2}^{N} \lambda_{max}(Z_i) \\
& + hr^2 \sum_{i=2}^{N} \lambda_{max}(Z_i) + h^2c^2r^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) \\
& + h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Sigma_i)\lambda_i^2\lambda_{max}^2(Z_i))\|w\| \\
& + 2\mu(t)\beta + 2hr \sum_{i=2}^{N} \lambda_{max}(Z_i)\mu(t)) + h\mu^2(t) \sum_{i=2}^{N} \lambda_i^2\lambda_{max}(Z_i). \qquad (5.40)
\end{aligned}
$$

Thus when

$$
\|w\| \geq \frac{2\mu(t)\beta + 2hr\sum_{i=2}^{N} \lambda_{max}(Z_i)\mu(t)}{\begin{array}{c} -(2\gamma\beta + 2hr\|D\bar{f}(s(t))\| \sum_{i=2}^{N} \lambda_{max}(Z_i) + 2h\mu(t)\|D\bar{f}(s(t))\| \sum_{i=2}^{N} \lambda_{max}(Z_i) \\ +hr^2 \sum_{i=2}^{N} \lambda_{max}(Z_i) + h^2c^2r^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) \\ +h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Sigma_i)\lambda_i^2\lambda_{max}^2(Z_i)) - \delta \end{array}},
$$

we have

$$
\dot{V} \leq -\delta\|w\|^2 + h\mu^2(t) \sum_{i=2}^{N} \lambda_{max}(Z_i)\lambda_i^2, \qquad (5.41)
$$

where Ξ is defined in (5.29). Thus, according to definition 4.1 and Lyapunov stability theory, bounded consensus is ultimately achieved.

Remark 5.2. The above two bounded consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical agents to the cases of non-identical agents. Because of the complexity of the consensus problems for non-identical agents, we only obtain here sufficient conditions instead of sufficient and necessary condition. At the same time, the conditions obtained here are somewhat complicated and difficult to verify, but according to certain specific cases, we can construct an appropriate numerical simulation example to verify the effectiveness of the proposed results. Comparing the above two theorems, it can be seen that the boundary of the convergence set and the maximum size of time delay can be evaluated respectively.

5.4 Controlled Bounded Consensus Criterion

We denote $x(t)$, $s(t)$, $u(t)$, $e(t)$, $w(t)$ and $V(w(t),t)$ as x, s, u, e, w and V respectively.

5.4.1 Linear Feedback Pinning Controllers

We apply the feedback control strategy on a small fraction δ $(0 < \delta \leq 1)$ of the agents in system (5.1). Suppose that nodes $i_1, i_2, \cdots, i_l$ are selected to be under control, where $l = [\delta N]$ stands for the smaller but nearest integer to the real number δN. This controlled NMAS can be described as

$$\begin{cases} \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j=1}^{N} a_{i_k j}\Gamma x_j(t-\tau) + u_{i_k}, & 1 \leq k \leq l, \\ \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j=1}^{N} a_{i_k j}\Gamma x_j(t-\tau), & l+1 \leq k \leq N. \end{cases} \tag{5.42}$$

The local linear negative feedback control law is chosen as follows:

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k} - s), & 1 \le k \le l, \\ u_{i_k} = 0, & l+1 \le k \le N, \end{cases} \tag{5.43}$$

where the feedback gain $d_{i_k} > 0$.

Combine (5.42) and (5.43) and rearrange the order of the nodes in the network. Let the first l agents be controlled, and $e_i = x_i - s,\ i = 1, 2, \cdots, N$. It's obvious that $\frac{c}{N}\sum_{k=1}^{N}\sum_{j=1}^{N} a_{kj}\Gamma x_j(t-\tau) = 0$ and $\sum_{i=1}^{N} e_i = 0$. Then by applying the Newton-Leibniz formula, error systems can be written as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t-\tau) + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s) - d_i e_i, 1 \le i \le l, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t-\tau) + \int_0^1 (Df_i + \tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s), l+1 \le i \le N. \end{cases} \tag{5.44}$$

The following work will focus on simplifying the error systems (5.44) by means of a series of transformations using a procedure similar to [88].

Define the following matrix

$$D = diag(D_1, D_2, \cdots, D_N) \in R^{nN \times nN},$$

where $D_i = diag\{-d_i, -d_i, \cdots, -d_i\} \in R^{n\times n}$.

Let $e = (e_1^T, e_2^T, \cdots, e_N^T)^T$, then (5.44) becomes

$$\dot{e} = \bar{\Sigma}(t)e + cA \otimes \Gamma e(t-\tau) + I(t)e - \frac{1}{N}H(t)e + F(t), \tag{5.45}$$

$\bar{\Sigma}(t) = I_N \otimes D\bar{f}(s) + D,\ H^T(t) = (H_1^T(t), \cdots, H_N^T(t)),\ H_i(t) = (\int_0^1 Df_1(s + \tau e_1)d\tau, \cdots, \int_0^1 Df_N(s + \tau e_N)d\tau),\ I(t) = diag\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau \cdots \int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\}$ and $F_i^T(t) = (f_1^T(s) - \bar{f}^T(s), \cdots, f_N^T(s) - \bar{f}^T(s))$ respectively.

Since A is symmetric and irreducible, according to [88], there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \cdots, \Phi_N)$. This together with $w(t) = (\Phi^T \otimes I_n)e$ gives

$$\begin{aligned}\dot{w} &= (\Phi^T \otimes I_n)\bar{\Sigma}(t)(\Phi \otimes I_n)w + (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)w(t-\tau)\\ &\quad + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w - \frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)w + (\Phi^T \otimes I_n)F(t).\end{aligned} \tag{5.46}$$

Note that $H(t) = \sqrt{N}\sum_{k=1}^N (\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its k-th column equal to Φ_1 and the remaining elements are zero. Then we have $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^N (\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau(\Phi \otimes I_n)$, where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the remaining of its elements are zero.

Then, $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \begin{pmatrix} \Upsilon_k \\ \mathsf{0} \end{pmatrix} \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k \in R^{1\times N}$ and $\mathsf{0} \in R^{(N-1)\times N}$. Therefore, $\dot{w} = \bar{\Sigma}(t)w + c\Lambda \otimes \Gamma w(t-\tau) + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w - \begin{pmatrix} * \\ \mathsf{0} \end{pmatrix} w + (\Phi^T \otimes I_n)F(t)$. Since $w_1 \equiv 0$, we only need to consider $w_2, w_3, \cdots, w_N$. Rewriting in the component form we have

$$\begin{aligned}\dot{w}_i &= \Sigma_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau) + (\Phi_i^T \otimes I_n)F(t)\\ &\quad + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w,\ i = 2, 3, \cdots, N,\end{aligned} \tag{5.47}$$

where $\Sigma_i = \bar{D}f(s) + D_i$.

So far, we have transferred the consensus problem of system (5.1) to the stability problem of the $N-1$ of $n-$dimensional systems.

Theorem 5.3 Suppose that $\|I(t)\| \leq \gamma$ is satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Theta_i > 0$, $\Pi_i > 0$, X_i, Y_i and Z_i of appropriate dimensions such that

$$B = \begin{pmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{pmatrix} < 0, \ \begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{5.48}$$

for $i = 2, 3, \cdots, N$, where $B_1 = \dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + hX_i + Y_i^T + Y_i + Q_i + h\Sigma_i^T(t)Z_i\Sigma_i(t)$, $B_2 = c\lambda_i P_i(t)\Gamma - Y_i + hc\lambda_i\Sigma_i^T(t)Z_i\Gamma$ and $B_3 = \Pi_i^{-1} + \Theta_i^{-1} - Q_i + hc^2\lambda_i^2\Gamma^T Z_i\Gamma$, then the NMAS (5.1) will achieve bounded consensus for the time-invariant delay $\tau \in [0, h]$ for some $h < \infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N}\sum_{k=1}^{3} V_k, \tag{5.49}$$

where

$$\begin{aligned} V_1 &= w_i^T P_i(t) w_i, \\ V_2 &= \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha d\beta, \\ V_3 &= \int_{t-\tau}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha. \end{aligned}$$

The i-th ($i = 2, 3, \cdots, N$) equation in system (5.47) can be written as

$$\begin{aligned} \dot{w}_i = (\Sigma_i(t) + c\lambda_i\Gamma)w_i - c\lambda_i\Gamma\int_{t-\tau}^{t}\dot{w}_i(\alpha)d\alpha \\ + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w + (\Phi_i^T \otimes I_n)F(t). \end{aligned} \tag{5.50}$$

Defining $a(.)$, $b(.)$ and M in [94] as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$ and $M = c\lambda_i P_i(t)\Gamma$ for all $\alpha \in [t-\tau, t]$ then we have

$$\begin{aligned}
\dot{V}_1 \leq\ & w_i^T[\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + hX_i + Y_i^T + Y_i]w_i \\
& + \int_{t-\tau}^{t} \dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha + 2w_i^T(c\lambda_i P_i(t)\Gamma - Y_i)w_i(t-\tau) \\
& + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t).
\end{aligned} \tag{5.51}$$

Moreover, $\dot{V}_2$ can be enlarged as

$$\begin{aligned}
\dot{V}_2 \leq\ & h[\Sigma_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau)]^T Z_i[\Sigma_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau)] \\
& + 2h(\Sigma_i(t)w_i)^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w + 2h(\Sigma_i(t)w_i)^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
& + 2h(c\lambda_i\Gamma w_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\
& + 2h(c\lambda_i\Gamma w_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
& + 2h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w)^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
& + h((\Phi_i^T \otimes I_n)F(t))^T Z_i((\Phi_i^T \otimes I_n)F(t)) \\
& + h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w)^T Z_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w) \\
& - \int_{t-\tau}^{t} \dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha.
\end{aligned} \tag{5.52}$$

and

$$\dot{V}_3 = w_i^T Q_i w_i - w_i^T(t-\tau)Q_i w_i(t-\tau). \tag{5.53}$$

Applying the Young Inequality, then $2h(c\lambda_i\Gamma w_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \leq w_i^T(t-\tau)\Pi_i^{-1}w_i(t-\tau) + h^2c^2\lambda_i^2 w^T((\Phi \otimes I_n)^T I(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Pi_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n))w(t)$, and $2h(c\lambda_i\Gamma w_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)F(t) \leq w_i^T(t-\tau)\Theta_i^{-1}w_i(t-\tau) + h^2c^2\lambda_i^2 F^T(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Theta_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)F(t)$. Applying these two inequalities and the conditions of the theorem results

$$
\begin{aligned}
\dot{V} \leq & \sum_{i=2}^{N} \begin{pmatrix} w_i \\ w_i(t-\tau) \end{pmatrix}^T B \begin{pmatrix} w_i \\ w_i(t-\tau) \end{pmatrix} \\
& + 2\mu(t)\beta + (\|w\|(2\gamma\beta + 2h\gamma\|\Sigma_i(t)\| \sum_{i=2}^{N} \lambda_{max}(Z_i) \\
& + h^2 c^2 \gamma^2 \lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Pi_i)\lambda_i^2 \lambda_{max}^2(Z_i) + 2h\mu(t)\|\Sigma_i(t)\| \sum_{i=2}^{N} \lambda_{max}(Z_i) \\
& + h^2 c^2 \mu^2(t) \lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Theta_i)\lambda_i^2 \lambda_{max}^2(Z_i))\|w\| + h\gamma^2 \sum_{i=2}^{N} \lambda_{max}(Z_i) \\
& + 2h\gamma \sum_{i=2}^{N} \lambda_{max}(Z_i)\mu(t)) + h\mu^2(t) \sum_{i=2}^{N} \lambda_i^2 \lambda_{max}(Z_i). \qquad (5.54)
\end{aligned}
$$

Thus when

$$
\|w\| \geq \frac{2\mu(t)\beta + 2h\gamma \sum_{i=2}^{N} \lambda_{max}(Z_i)\mu(t)}{\varpi(t)},
$$

we have

$$
\dot{V} \leq -\delta\|w\|^2 + h\mu^2(t) \sum_{i=2}^{N} \lambda_{max}(Z_i)\lambda_i^2, \qquad (5.55)
$$

where, $\varpi(t) = -2\gamma\beta - 2h\gamma\|\Sigma_i(t)\| \sum_{i=2}^{N} \lambda_{max}(Z_i) - 2h\mu(t)\|\Sigma_i(t)\| \sum_{i=2}^{N} \lambda_{max}(Z_i) - h\gamma^2 \sum_{i=2}^{N} \lambda_{max}(Z_i) - h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) - h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}} (\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Theta_i)\lambda_i^2\lambda_{max}^2(Z_i) - \delta$. Thus, according to [92] and Lyapunov stability theory, bounded consensus is ultimately achieved. This completes the proof.

5.4.2 Adaptive Pinning Controllers

Assume that the first l agents are selected as pinned agents with the adaptive controllers:

$$\begin{cases} u_i = -d_i(t)(x_i - s), & 1 \le i \le l, \\ \dot{d}_i(t) = h_i e_i^T P_i(t) e_i, & \\ u_i = 0, & l+1 \le i \le N, \end{cases} \tag{5.56}$$

where constant $h_i > 0$ and positive definite matrix $P_i(t) \in R^{n\times n}$. Applying Newton-Leibniz formula, then the error NMAS can be rewritten as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t-\tau) + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s) - d_i(t)e_i, \le i \le l, \\ \dot{d}_i(t) = h_i e_i^T P_i(t) e_i, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t-\tau) + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s), l+1 \le i \le N. \end{cases} \tag{5.57}$$

Repeating a similar procedure to the previous subsection, then we have

$$\begin{cases} \dot{w}_i = D\bar{f}(s(t))w_i - d_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \qquad +(\Phi_i^T \otimes I_n)F(t), \qquad 2 \le i \le l, \\ \dot{d}_i(t) = h_i w_i^T P_i(t) w_i, \\ \dot{w}_i = D\bar{f}(s)w_i + c\lambda_i\Gamma w_i(t-\tau) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \qquad +(\Phi_i^T \otimes I_n)F(t), \qquad l+1 \le i \le N, \end{cases} \tag{5.58}$$

where w_i, w, Φ, Φ_i, $I(t)$ and $F(t)$ are the same as the previous subsection.

Theorem 5.4 Suppose that $\|I(t)\| \leq \gamma$ is satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Theta_i > 0$, $\Pi_i > 0$, X_i, Y_i and Z_i of appropriate dimensions and constant $d > 0$ such that

$$B = \begin{pmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{pmatrix} < 0, \begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{5.59}$$

for $i = 2, 3, \cdots, N$, where $B_1 = \dot{P}_i(t) + P_i(t)(Df(s)) + (Df(s))^T P_i(t) - 2dP_i(t) + hX_i + Y_i^T + Y_i + Q_i + h\Sigma_i^T(t)Z_i\Sigma_i(t)$, $B_2 = c\lambda_i P_i(t)\Gamma - Y_i + hc\lambda_i\Sigma_i^T(t)Z_i\Gamma$ and $B_3 = \Pi_i^{-1} + \Theta_i^{-1} - Q_i + hc^2\lambda_i^2\Gamma^T Z_i\Gamma$, then the system (5.1) will achieve bounded consensus for the time-invariant delay $\tau \in [0, h]$ for some $h < \infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N}\sum_{k=1}^{3} V_k + \sum_{i=2}^{l} \frac{(d_i(t) - d)^2}{h_i}, \tag{5.60}$$

where

$$\begin{aligned} V_1 &= w_i^T P_i(t) w_i, \\ V_2 &= \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha d\beta, \\ V_3 &= \int_{t-\tau}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha. \end{aligned}$$

The remaining part of the proof is similar to that of Theorem 5.1 and is therefore omitted here. This completes the proof.

5.5 Examples

In this section, we will construct examples to demonstrate the proposed results above.

Example 5.1 The agent dynamics can be chosen as follows

$$\dot{x}_i(t) = B_i x_i(t) + g(x_i(t)), \quad i = 1, 2, \cdots, 11, \tag{5.61}$$

where

$$\begin{cases} B_i = \begin{pmatrix} -10 + 0.1 \times (i-1) & 10 - 0.1 \times (i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15 - 0.1 \times (i-1) & 0 \end{pmatrix}, \quad i = 1, 2, \cdots, 6, \\ B_i = \begin{pmatrix} -10 - 0.1 \times (i-6) & 10 + 0.1 \times (i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15 + 0.1 \times (i-6) & 0 \end{pmatrix}, \quad i = 7, 8, \cdots, 11, \end{cases}$$

and

$$g(x_i(t)) = (-9.5 sin(\frac{\pi x_{i1}(t)}{3.2} + \pi) \ 0 \ 0)^T, \quad i = 1, 2, \cdots, 11.$$

The communication coupling matrix A and the inner coupling matrix are

$$A = \begin{pmatrix} -8 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -8 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -6 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & -7 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -5 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -10 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -6 \end{pmatrix} \Gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

respectively.

Given the initial values of 11 agents as $(10\ 5\ -10)^T$, $(12\ 6\ -12)^T$, $(14\ 7\ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$ and $(-10\ 15\ 10)^T$ respectively. We may verify the conditions of Theorem 5.1 and Theorem 5.2 readily. This demonstrates the consensus of the NMAS is achieved for any time delay $0 < \tau \leq 0.061$. Simulation results are depicted in Fig 5.1 to Fig 5.5 for $\tau = 0.061$ and $c = 1$.

The simulation curves in Fig 5.1 show that the states of all agents are ultimately bounded stable. The average state trajectory $s(t)$ is chosen as the desired moving trajectory and is depicted in Fig 5.2 Fig 5.3 to Fig 5.5 demonstrate that the state errors between each agent's states and the desired state trajectory respectively, and the deviation systems are also ultimately bounded stable. These simulation curves show that all agents eventually move with the desired state trajectory in the sense of boundedness.

Example 5.2

The controlled consensus results will be verified based on the NMAS (5.61).

Design the following controllers

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k}(t) - s(t)), & i_k = 1, 2 \texttt{ and } 10, \\ u_{i_k} = 0, & \texttt{else}, \end{cases}$$

with $d_1 = 0.5$, $d_2 = 0.5$, $d_{10} = 0.5$ and

$$\begin{cases} u_{i_k} = -d_{i_k}(t)(x_{i_k}(t) - s(t)), & i_k = 1, 2 \texttt{ and } 10, \\ \dot{d}_{i_k}(t) = h_{i_k} e_{i_k}^T P_{i_k}(t) e_{i_k}, & \\ u_{i_k} = 0, & \texttt{else}, \end{cases}$$

with $h_1 = 0.1$, $h_2 = 0.2$, $h_{10} = 0.3$, $s(t)$ can then be evaluated by simulation.

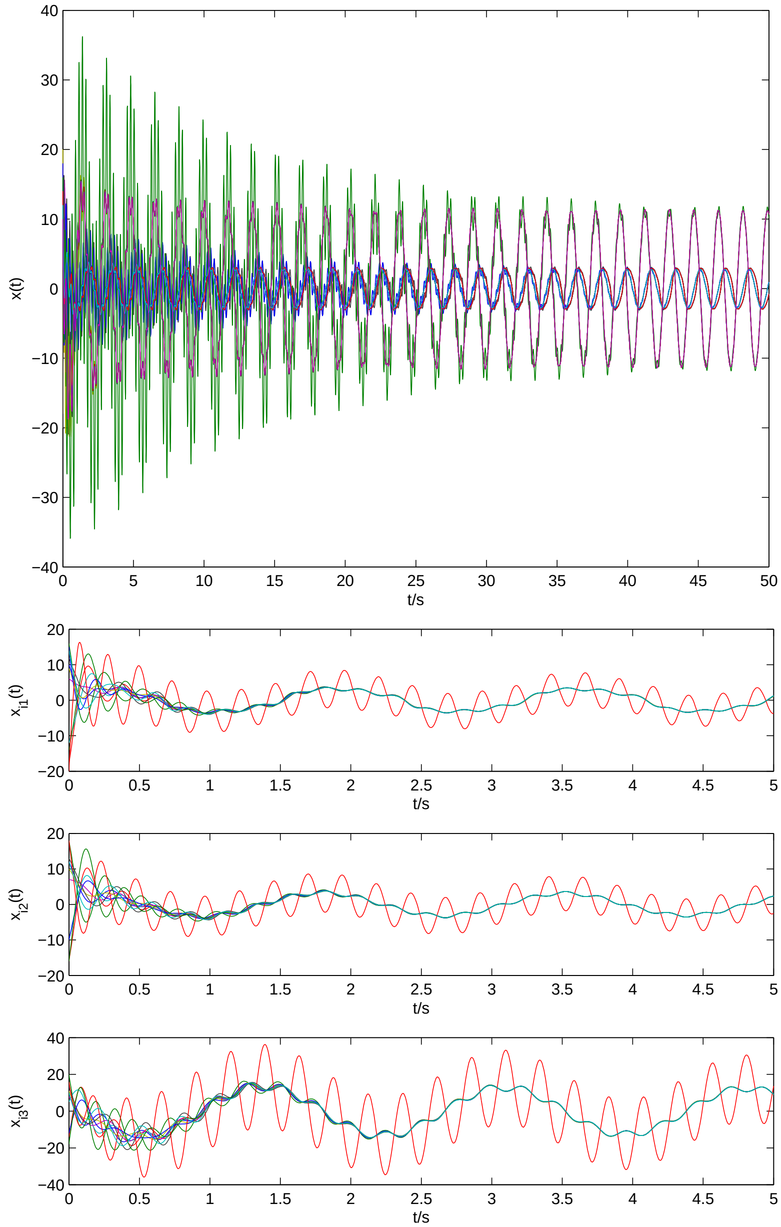

Fig 5.1 The dynamics of all agents.

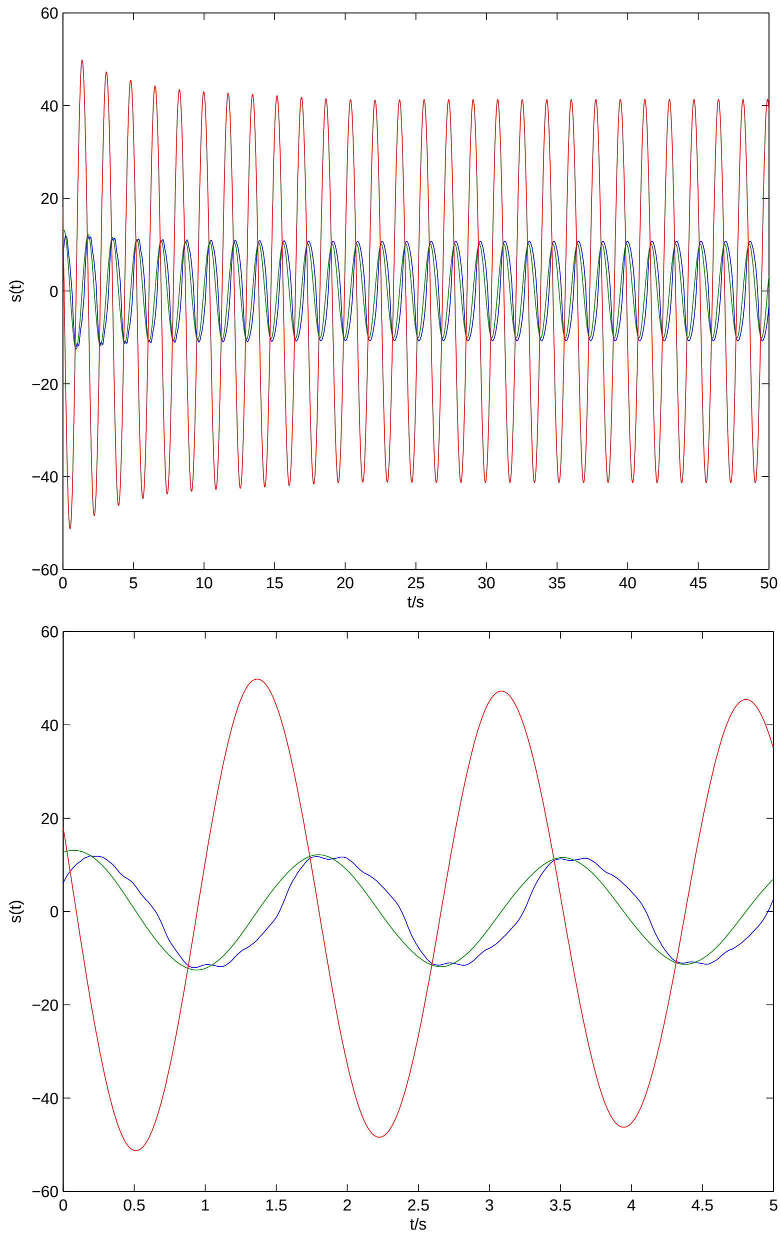

Fig 5.2 The average state trajectory $s(t)$.

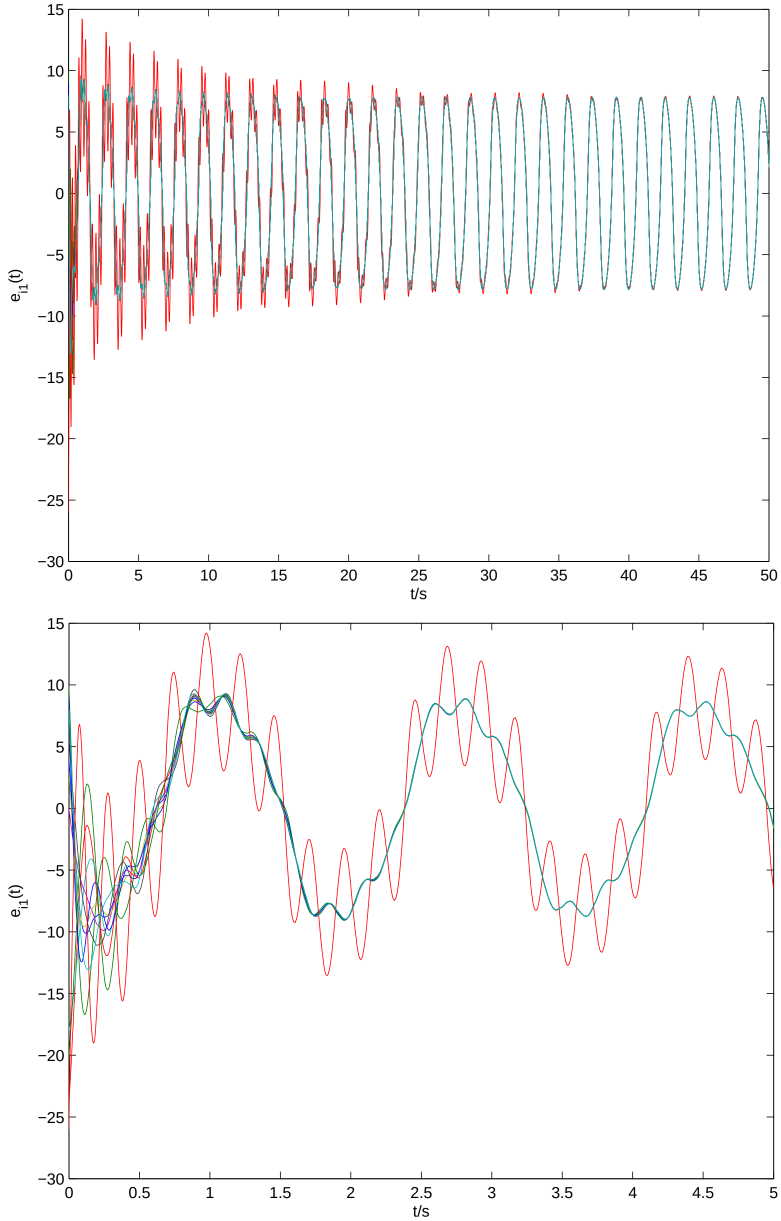

Fig 5.3 The consensus error dynamics for the first dynamic of each agent.

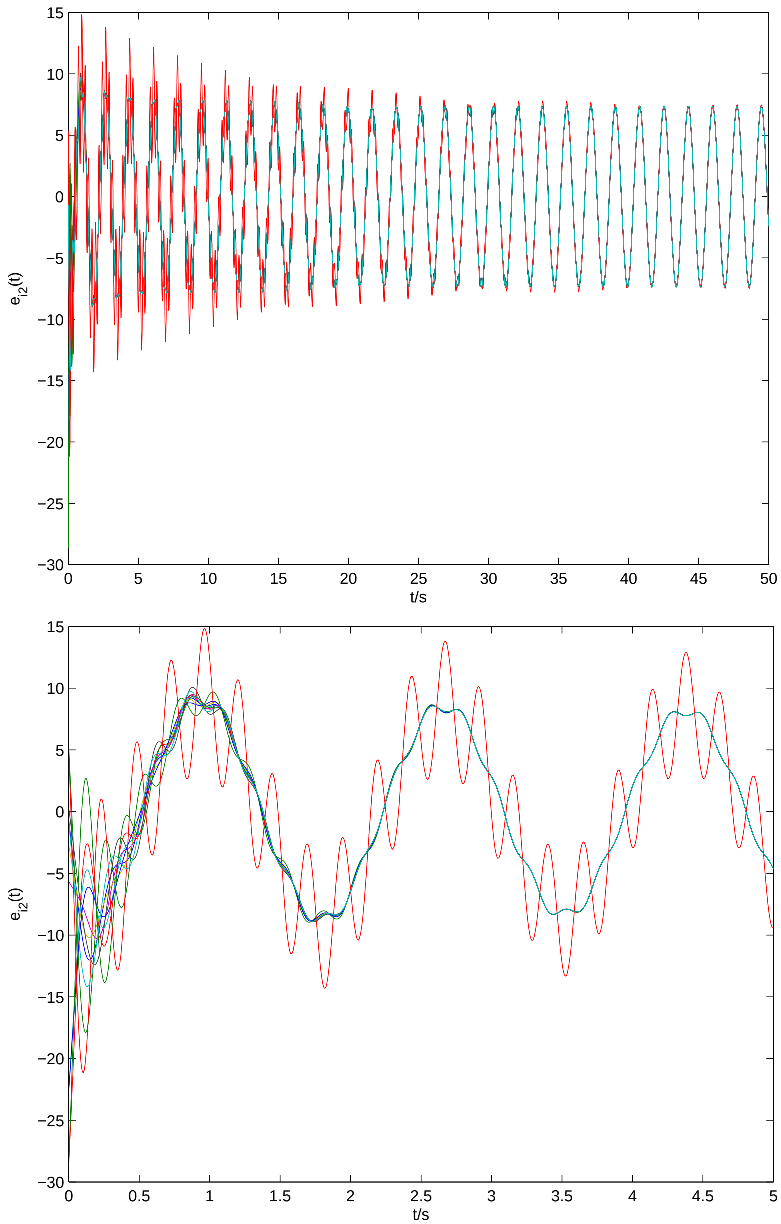

Fig 5.4 The consensus error dynamics for the second dynamic of each agent.

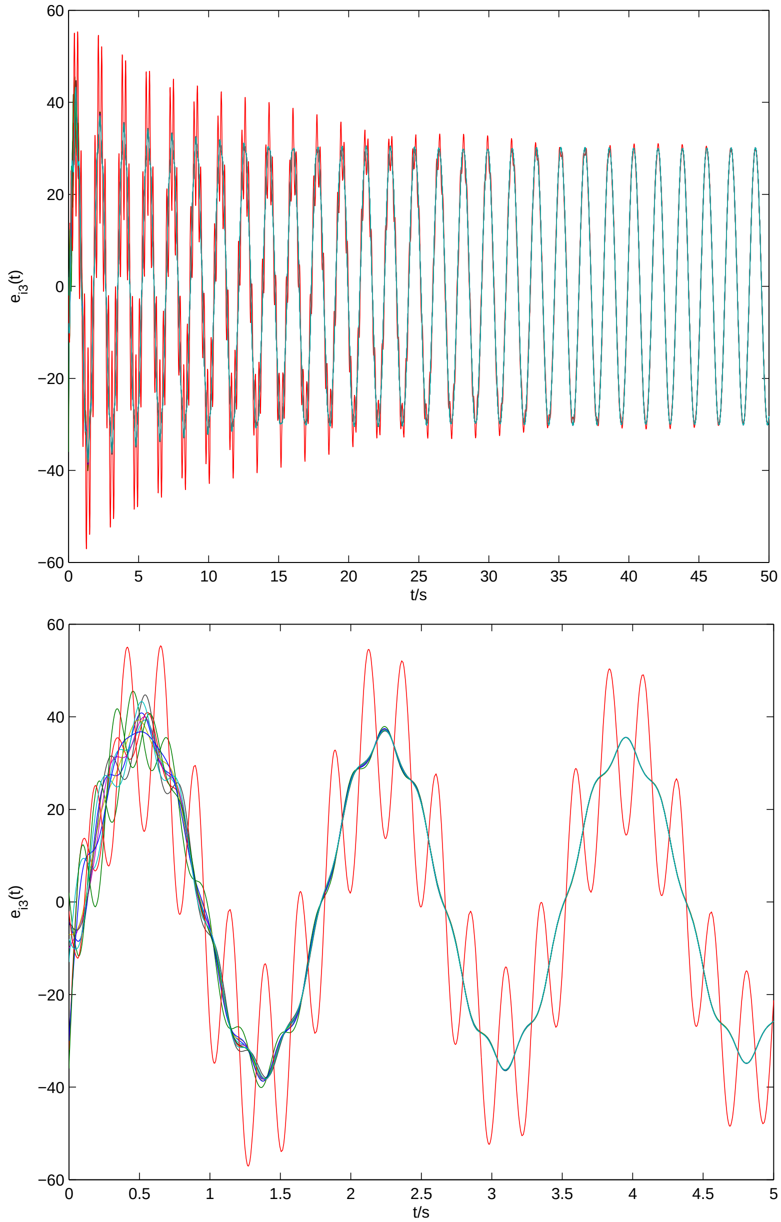

Fig 5.5 The consensus error dynamics for the third dynamic of each agent.

Given the initial values of 11 agents as $(10\ 5\ \ -10)^T$, $(12\ 6\ \ -12)^T$, $(14\ 7\ \ -14)^T$, $(16\ \ 8\ \ \ -16)^T$, $(18\ \ 9\ \ \ -18)^T$, $(20\ \ 10\ \ \ -20)^T$, $(-18\ \ 11\ \ 18)^T$, $(-16\ \ 12\ \ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$, $(-10\ 15\ 10)^T$ respectively and $P_{i_k}(t) = I_3$. We may verify the conditions of Theorem 5.3 and Theorem 5.4 readily. This demonstrates the bounded consensus of the MAS is achieved for any time delay $0 < \tau \leq 0.061$. Simulation results are depicted in Fig 5.6 to Fig 5.9 for $\tau = 0.061$ and $c = 1$.

5.6 Conclusions

In this chapter, we've investigated the consensus problems and controlled consensus problems of NMAS with different agent dynamics. The derived results are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on a series of transformations and Lyapunov stability theorem. The methods we presented here have several distinct features. Firstly, they are very simple in form, but are more effective to resolve the consensus problem with non-identical node dynamics. Secondly, the communication connection between agents are not direct, and there are constant time delays in the communication topology. It should be noted that the conditions are still restrictive and all the delays are the same, and further investigations will focus on relaxing these limitations.

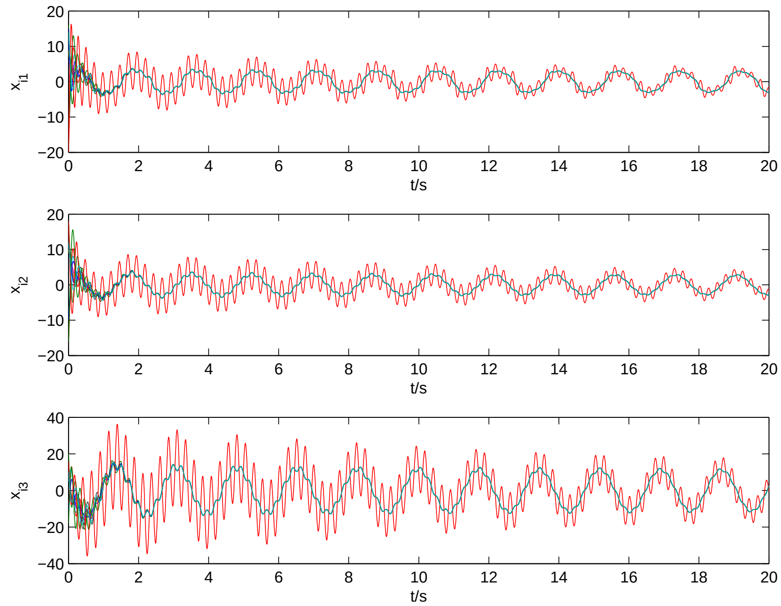

Fig 5.6 All agent dynamics under pinning control.

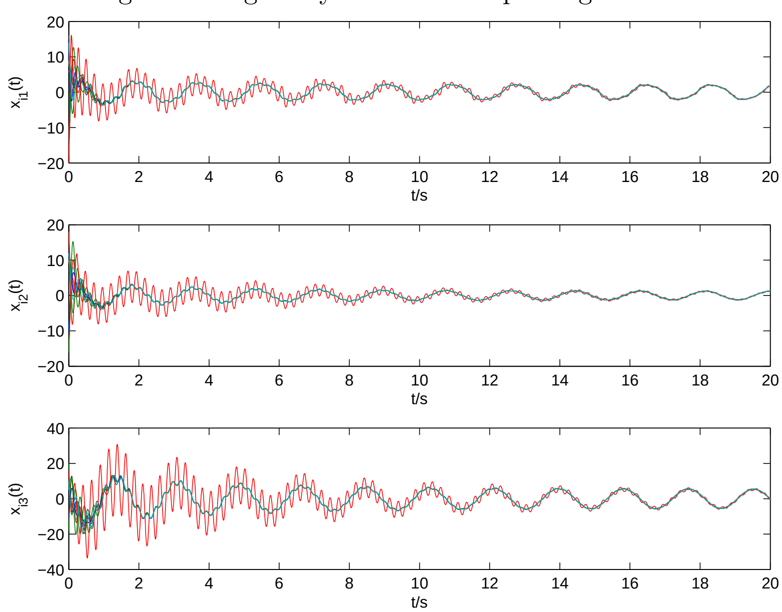

Fig 5.7 All agent dynamics under adaptive pinning control.

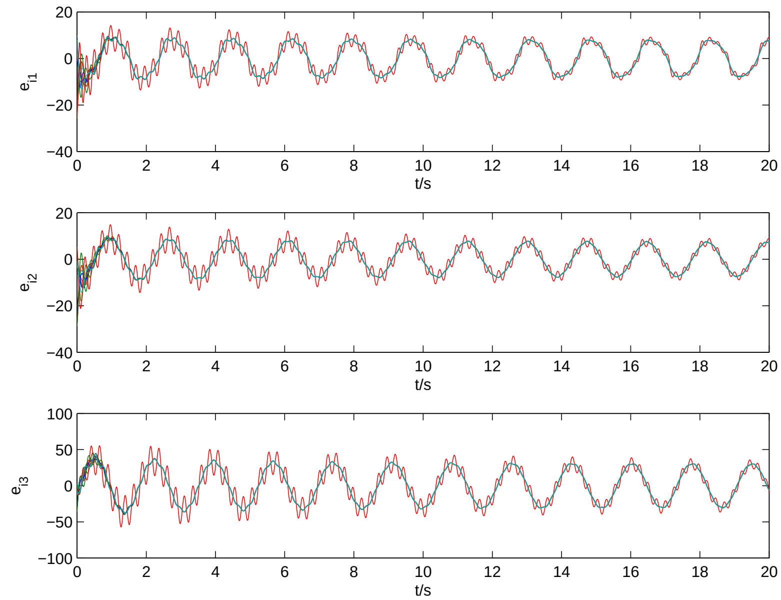

Fig 5.8 All agent error dynamics under pinning control.

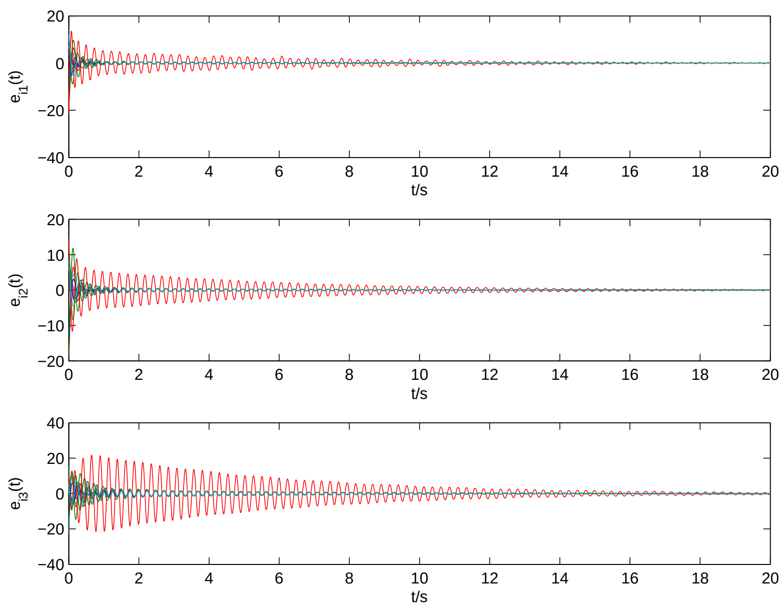

Fig 5.9 All agent error dynamics under adaptive pinning control.

Chapter 6

Global Consensus of NMAS with Different Agent Dynamics and Time-Varying Delay Topology

The global bounded consensus problem of NMAS exhibiting nonlinear, non-identical agent dynamics with communication time-varying delay will be investigated in this chapter. Delay-independent and delay-dependent bounded consensus criterion and controlled bounded consensus criterion based on the Lypunov-Krasovskii functional method and pinning control scheme are derived. The proposed consensus criteria ensures that all agents eventually move along the desired trajectories in the sense of boundedness. Many related results in this area can be viewed as special cases of the above proposed consensus criterion. The effectiveness of the theoretical results are verified by means of numerical simulations.

6.1 Introduction

The present chapter will extend the results in the previous chapter to the time-varying coupling case. This chapter will focus on the global consensus problems of NMAS based on pinning control methods, and the proposed consensus criterion and controlled consensus criterion is formulated in terms of certain boundedness of state errors.

The rest of this chapter is organized as follows. A continuous-time NMAS model with time-varying communication delay is presented in section 6.2 and its consensus analysis is made in section 6.3. In section 6.4, pinning control and adaptive pinning control are introduced in and controlled bounded consensus criterion is derived in section 6.4. Numerical simulation examples are given in section 6.5 to verify the effectiveness of the proposed results, followed by conclusions in section 6.6.

6.2 Problem Description

Consider a NMAS consisting of N non-identical agents with time-varying communication delay:

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma x_j(t-\tau(t)), \quad i = 1, 2, \cdots, N, \tag{6.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i(t)) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix, and where $\gamma_{ij} \neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N\times N}$(which is symmetric and irreducible) represents the communication topology relation of the NMAS, and is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j\neq i} a_{ij}$. $\tau(t)$ is a time-varying coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field

$$\bar{f}(x(t)) = \frac{1}{N}\sum_{k=1}^{N} f_k(x(t)). \tag{6.2}$$

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t). \tag{6.3}$$

We now discuss the problem of global consensus for the NMAS (6.1). The consensus problem formulation here is the same as in the previous chapter, i.e., the consensus problem is solvable if the states of all agents satisfy certain boundedness of $x_i(t) - x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. .

6.3 Global Bounded Consensus Analysis

Define the error vector

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \cdots, N. \tag{6.4}$$

Obviously, $\sum_{i=i}^{N} e_i = 0$ and $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i}^{N} a_{kj}\Gamma x_j(t-\tau(t)) = 0$, then the NMAS (6.1) can be rewritten in terms of e_i as

$$\dot{e}_i(t) = f_i(s(t) + e_i(t)) - \frac{1}{N}\sum_{k=1}^{N} f_k(s(t) + e_k(t)) + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma e_j(t-\tau(t)). \tag{6.5}$$

The following work will focus on simplifying the error NMAS (6.5) by means of a series of transformations using a procedure similar to [88].

Applying the Newton-Leibniz formula, error NMAS (6.5) can be further written as

$$
\begin{aligned}
\dot{e}_i(t) = {} & D\bar{f}(s(t))e_i(t) + c\sum_{j\in\mathbb{N}_i}^{N} a_{ij}\Gamma e_j(t-\tau(t)) \\
& + \int_0^1 (Df_i(s(t)+\tau e_i(t)) - D\bar{f}(s(t)))e_i(t)d\tau \\
& - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s(t)+\tau e_k(t))e_k(t)d\tau + f_i(s(t)) - \bar{f}(s(t)). \qquad (6.6)
\end{aligned}
$$

If we consider the linearized NMAS of (6.1), we have

$$
\begin{aligned}
\dot{e}_i(t) = {} & D\bar{f}(s(t))e_i(t) + c\sum_{j\in\mathbb{N}_i}^{N} a_{ij}\Gamma e_j(t-\tau(t)) + (Df_i(s(t)) - D\bar{f}(s(t)))e_i(t) \\
& - \frac{1}{N}\sum_{k=1}^{N} Df_k(s(t))e_k(t) + f_i(s(t)) - \bar{f}(s(t)). \qquad (6.7)
\end{aligned}
$$

Let $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$, then (6.6) becomes

$$
\dot{e}(t) = I_N \otimes D\bar{f}(s)e(t) + cA\otimes\Gamma e(t-\tau(t)) + I(t)e(t) - \frac{1}{N}H(t)e(t) + F(t), \qquad (6.8)
$$

where

$$
\begin{aligned}
I(t) = diag\{ & \int_0^1 (Df_1(s(t)+\tau e_1(t)) - D\bar{f}(s(t)))d\tau \cdots \\
& \int_0^1 (Df_N(s(t)+\tau e_N(t)) - D\bar{f}(s(t)))d\tau\},
\end{aligned}
$$

$$H(t)=\begin{pmatrix} \int_0^1 Df_1(s(t)+\tau e_1(t))d\tau & \cdots & \int_0^1 Df_N(s(t)+\tau e_N(t))d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s(t)+\tau e_1(t))d\tau & \cdots & \int_0^1 Df_N(s(t)+\tau e_N(t))d\tau \end{pmatrix},$$

$$F(t)=\begin{pmatrix} f_1(s(t))-\bar{f}(s(t)) \\ \vdots \\ f_N(s(t))-\bar{f}(s(t)) \end{pmatrix}.$$

Since A is symmetric and irreducible, according to Lemma 4.1, there exists a unitary matrix $\Phi=(\varphi_{ij})_{N\times N}=(\Phi_1,\Phi_2,\cdots,\Phi_N)$, such that (**??**) is satisfied. This together with $\omega(t)=(\Phi^T\otimes I_n)e(t)$ gives

$$\begin{aligned}
&\dot{\omega}(t)=(\Phi^T\otimes I_n)\dot{e}(t)\\
&=(\Phi^T\otimes I_n)(I_N\otimes D\bar{f}(s))(\Phi\otimes I_n)\omega(t)+(\Phi^T\otimes I_n)(cA\otimes\Gamma)(\Phi\otimes I_n)\omega(t-\tau(t))\\
&+(\Phi^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)-\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)\omega(t)+(\Phi^T\otimes I_n)F(t).
\end{aligned} \tag{6.9}$$

Note that

$$H(t)=\sqrt{N}\sum_{k=1}^{N}\left[(\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0})\otimes\int_0^1 Df_k(s(t)+\tau e_k(t))d\tau\right], \tag{6.10}$$

where $\bar{\Phi}_k$ stands for the matrix with its k-th column equals Φ_1 and the rest of its elements are zero, then we have

$$\begin{aligned}
\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)=\frac{1}{\sqrt{N}}&\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0})\\
&\otimes\int_0^1 Df_k(s(t)+\tau e_k(t))d\tau(\Phi\otimes I_n),
\end{aligned} \tag{6.11}$$

where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the rest of its elements are zero.

Thus, a simple calculation gives

$$\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \begin{pmatrix} \Upsilon_k \\ \mathsf{0} \end{pmatrix} \otimes \int_0^1 Df_k(s(t) + \tau e_k(t))d\tau, \tag{6.12}$$

where $\Upsilon_k \in R^{1\times N}$ and $\mathsf{0} \in R^{(N-1)\times N}$.

Therefore,

$$\begin{aligned} \dot{\omega}(t) = &\, I_N \otimes D\bar{f}(s)\omega(t) + c\Lambda \otimes \Gamma\omega(t-\tau(t)) + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\ &- \begin{pmatrix} * \\ \mathsf{0} \end{pmatrix} \omega(t) + (\Phi^T \otimes I_n)F(t). \end{aligned} \tag{6.13}$$

Since $\omega_1 \equiv 0$, we only need to consider $\omega_2(t), \omega_3(t), \cdots, \omega_N(t)$. Rewriting (6.13) in the component form we have

$$\begin{aligned} \dot{\omega}_i(t) = &\, D\bar{f}(s(t)\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau(t)) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\ &+ (\Phi_i^T \otimes I_n)F(t),\ i = 2, 3, \cdots, N. \end{aligned} \tag{6.14}$$

So far, we have transferred the consensus problem of NMAS (6.1) to the stability problem of the $N-1$ of $n-$dimensional systems.

In the following, a time-independent global consensus criteria is derived for the NMAS (6.1).

Theorem 6.1 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i

and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$a\|x(t)\|^2 \leq x^T(t)P_i(t)x(t) + \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \leq b\|x(t)\|^2,$$
$$\forall t \in R^+,\ x \in R^n, i = 2, 3, \cdots, N, \tag{6.15}$$

$$\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + Q_i + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t)$$
$$+ \zeta I \leq 0,\ i = 1, 2, \cdots, N, \tag{6.16}$$

$$\|I(t)\| \leq \gamma,\ i = 1, 2, \cdots, N. \tag{6.17}$$

Then the system (6.43) converges to the set

$$M = \{e(t)|\|e(t)\| \leq \frac{2b}{a}\frac{\beta\overline{lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\} \tag{6.18}$$

for any fixed time delay $\tau > 0$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, $\beta = (\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}}$, $\zeta > 2\gamma\beta$ and $\mu(t) = \|F(t)\|$ is bounded. Furthermore, the NMAS (6.1) achieves bounded consensus for any fixed time delay $\tau(t) > 0$, $0 \leq \dot{\tau}(t) \leq 1$.

Proof. Choose the following Lyapunov-Krasovskii functional as

$$V(w_i(t), t) = \sum_{i=2}^{N} V_i(w_i(t), t), \tag{6.19}$$

$$V_i(w_i(t), t) = w_i^T(t)P_i(t)w_i(t) + \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha, \quad i = 2, 3, \cdots, N. \tag{6.20}$$

Differentiating (6.53) along the trajectory of (6.45) gives

$$\dot{V}_i(w_i(t),t) = w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + Q_i)w_i(t)$$
$$+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t)$$
$$+ 2w_i^T(t)(c\lambda_i P_i(t)\Gamma)w_i(t-\tau(t)) - (1-\dot{\tau}(t))w_i^T(t-\tau(t))Q_i w_i(t-\tau(t)). \tag{6.21}$$

Applying the Young Inequality to the equality (6.54), results in

$$\dot{V}_i(w_i(t),t) \le w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t)$$
$$+ Q_i + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i(t)$$
$$+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t). \tag{6.22}$$

Condition (6.66) implies that the first term on the right hand side of (6.55) satisfies

$$w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t)$$
$$+ Q_i + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i(t) \le -\zeta\|w_i(t)\|^2. \tag{6.23}$$

Applying condition (6.48) we know the second term on the right hand side of (6.55) satisfies

$$2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \le 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|. \tag{6.24}$$

The third term on the right hand side of (6.55) satisfies

$$2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \le 2\mu(t)\|P_i(t)\|\|w_i(t)\|. \tag{6.25}$$

Since $V(w(t),t) = \sum_{i=2}^{N} V_i(w_i(t),t)$, we have

$$
\begin{aligned}
\dot{V}(w(t),t) &= \sum_{i=2}^{N} \dot{V}_i(w_i(t),t) \\
&\le \sum_{i=2}^{N}(-\zeta\|w_i(t)\|^2) + 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\| + 2\mu(t)\|P_i(t)\|\|w_i(t)\| \\
&= -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\| + \mu(t))\sum_{i=2}^{N}\|w_i(t)\|\|P_i(t)\| \\
&\le -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\| + \mu(t))\|w(t)\|(\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}} \\
&= \|w(t)\|((2\gamma\beta - \zeta)\|w(t)\| + 2\beta\mu(t)). \qquad (6.26)
\end{aligned}
$$

Thus, when

$$\|w(t)\| \ge \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \qquad (6.27)$$

we have

$$\dot{V}(w(t),t) \le -\delta\|w(t)\|^2. \qquad (6.28)$$

Applying Lemma 4.2 completes the proof.

Corollary 6.1 We have an asymptotic consensus criterion in the classical sense when $\overline{lim}_{t\to\infty}\mu(t) = 0$. In particular, we have $\mu(t) \equiv 0$ when all agents have the same dynamics, i.e., $f_i(x_i(t)) = f(x(t))$. In such a case, applying theorem 6.1 to the linearized network (6.7), which is equivalent to taking $\gamma = 0$ in (6.48), immediately achieves the universal consensus criteria existing in many literatures. Therefore, Theorem 6.1 covers the existing criteria of networks with identical agent dynamics as a special case.

Remark 6.1 The above result is a delay-independent globally consensus criterion and the ultimate convergence bound is evaluated by means of (6.51). Theorem 3.1 guarantees that all agents move along the desired trajectory described by $s(t)$ in terms of certain boundedness, i.e., the consensus achieved here is just approximate instead of exact, in fact, to achieve exact consensus is impossible for such a case.

Next, we will provide delay-dependent criterion for the proposed problem.

Theorem 6.2 Suppose that (6.46) and (6.48) in Theorem 6.1 are satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Pi_i > 0$, $\Pi_i > 0$, X_i, Y_i and Z_i of appropriate dimensions such that

$$\Xi = \begin{pmatrix} \Xi_{11} & c\lambda_i P_i(t)\Gamma - Y_i + hc\lambda_i D\bar{f}(s(t))^T Z_i \Gamma \\ * & \Pi_i^{-1} + \Sigma_i^{-1} - Q_i + hc^2\lambda_i^2 \Gamma^T Z_i \Gamma \end{pmatrix} < 0, \tag{6.29}$$

where $\Xi_{11} = \dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + hX_i + Y_i^T + Y_i + Q_i + hD\bar{f}(s(t))^T Z_i D\bar{f}(s(t))$ and

$$\begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{6.30}$$

for $i = 2, 3, \cdots, N$, then the NMAS (6.1) will achieve bounded consensus for the time varying delay $0 \leq \tau(t) \leq h < \infty$, $0 < \dot{\tau}(t) \leq 1$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V(w_i(t), t) = \sum_{i=2}^{N} \sum_{k=1}^{3} V_k(w_i(t), t), \tag{6.31}$$

where

$$
\begin{aligned}
V_1(w_i(t),t) &= w_i^T(t)P_i(t)w_i(t),\\
V_2(w_i(t),t) &= \int_{-\tau(t)}^{0}\int_{t+\beta}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha d\beta,\\
V_3(w_i(t),t) &= \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_iw_i(\alpha)d\alpha.
\end{aligned}
$$

The i-th ($i = 2,3,\cdots,N$) equation in system (6.45) can be written as

$$
\begin{aligned}
\dot{\omega}_i(t) &= (D\bar{f}(s(t)) + c\lambda_i\Gamma)\omega_i(t) - c\lambda_i\Gamma\int_{t-\tau(t)}^{t}\dot{w}_i(\alpha)d\alpha\\
&\quad + (\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t) + (\Phi_i^T\otimes I_n)F(t), \quad i = 2,3,\cdots,N, \quad (6.32)
\end{aligned}
$$

and, thus, the derivative of $V_1(w_i(t),t)$ satisfies

$$
\begin{aligned}
\dot{V}_1(w_i(t),t) &= w_i^T(t)(\dot{P}_i(t) + P_i(t)(D\bar{f}(s(t)) + c\lambda_i\Gamma)\\
&\quad + (D\bar{f}(s(t)) + c\lambda_i\Gamma)^TP_i(t))w_i(t) - 2c\lambda_iw_i^T(t)P_i(t)\Gamma\int_{t-\tau(t)}^{t}\dot{w}_i(\alpha)d\alpha\\
&\quad + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w(t)\\
&\quad + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)F(t). \quad (6.33)
\end{aligned}
$$

Defining $a(.)$, $b(.)$ and M as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$ and $M = c\lambda_iP_i(t)\Gamma$ for all $\alpha\in[t-\tau(t),t]$ and then applying Lemma 5.1 results in

$$
\begin{aligned}
\dot{V}_1(w_i(t),t) &\leq w_i^T(t)[\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^TP_i(t) + hX_i + Y_i^T\\
&\quad + Y_i]w_i(t) + 2w_i^T(t)(c\lambda_iP_i(t)\Gamma - Y_i)w_i(t-\tau(t)) + \int_{t-\tau(t)}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha\\
&\quad + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)F(t).
\end{aligned}
\quad (6.34)
$$

Moreover, since

$$\dot{V}_2(w_i(t),t) = \tau(t)[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau(t)) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) + (\Phi_i^T \otimes I_n)F(t)]^T Z_i[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau(t)) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) + (\Phi_i^T \otimes I_n)F(t)] - \int_{t-\tau(t)}^{t} \dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha, \tag{6.35}$$

the above equality can be enlarged as

$$\begin{aligned}
&\dot{V}_2(w_i(t),t) \\
&\quad \le h[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau(t))]^T Z_i[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau(t))] \\
&\quad + 2h(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\
&\quad + 2h(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
&\quad + 2h(c\lambda_i\Gamma\omega_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\
&\quad + 2h(c\lambda_i\Gamma\omega_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
&\quad + 2h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
&\quad + h((\Phi_i^T \otimes I_n)F(t))^T Z_i((\Phi_i^T \otimes I_n)F(t)) \\
&\quad + h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)) \\
&\quad - \int_{t-\tau(t)}^{t} \dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha.
\end{aligned} \tag{6.36}$$

and

$$\dot{V}_3(w_i(t),t) = w_i^T(t)Q_iw_i(t) - (1-\dot{\tau}(t))w_i^T(t-\tau(t))Q_iw_i(t-\tau(t)). \tag{6.37}$$

The derivative of $V(w_i(t), t)$ is

$$
\begin{aligned}
&\sum_{k=1}^{3} \dot{V}_k(w_i(t), t) \\
&\le w_i^T(t)[\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + hX_i + Y_i^T + Y_i]w_i(t) \\
&\ + 2w_i^T(t)(c\lambda_i P_i(t)\Gamma - Y_i)w_i(t-\tau(t)) \\
&\ + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \\
&\ + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \\
&\ + 2h(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\
&\ + h[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau(t))]^T Z_i[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau(t))] \\
&\ + 2h(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
&\ + 2h(c\lambda_i\Gamma\omega_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\
&\ + 2h(c\lambda_i\Gamma\omega_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
&\ + 2h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\
&\ + h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)) \\
&\ + h((\Phi_i^T \otimes I_n)F(t))^T Z((\Phi_i^T \otimes I_n)F(t)) + w_i^T(t)Q_i w_i(t) \\
&\ - (1 - \dot{\tau}(t))w_i^T(t-\tau(t))Q_i w_i(t-\tau(t)).
\end{aligned}
$$

Applying the Young Inequality, we have $2h(c\lambda_i\Gamma\omega_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \le h^2c^2\lambda_i^2 w^T(t)((\Phi \otimes I_n)^T I(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Pi_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n))w(t) + w_i^T(t-\tau(t))\Pi_i^{-1}w_i(t-\tau(t))$ and $2h(c\lambda_i\Gamma\omega_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)F(t) \le w_i^T(t-\tau(t))\Sigma_i^{-1}w_i(t-\tau(t)) + h^2c^2\lambda_i^2 F^T(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Sigma_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)F(t)$.

Applying (6.46) and (6.48) to the above inequality results in:

$$
\begin{aligned}
\dot{V}(w_i(t),t) \leq & \sum_{i=2}^{N} \begin{pmatrix} w_i(t) \\ w_i(t-\tau(t)) \end{pmatrix}^T \Xi \begin{pmatrix} w_i(t) \\ w_i(t-\tau(t)) \end{pmatrix} \\
& + \|w\|((2\gamma\beta + 2hr\|D\bar{f}(s(t))\| \sum_{i=2}^{N} \lambda_{max}(Z_i) \\
& + 2h\mu(t)\|D\bar{f}(s(t))\| \sum_{i=2}^{N} \lambda_{max}(Z_i) \\
& + hr^2 \sum_{i=2}^{N} \lambda_{max}(Z_i) + h^2c^2r^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) \\
& + h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Sigma_i)\lambda_i^2\lambda_{max}^2(Z_i))\|w\| \\
& + 2\mu(t)\beta + 2hr \sum_{i=2}^{N} \lambda_{max}(Z_i)\mu(t)) + h\mu^2(t) \sum_{i=2}^{N} \lambda_i^2\lambda_{max}(Z_i). \qquad (6.38)
\end{aligned}
$$

Thus when

$$
\|w\| \geq \frac{2\mu(t)\beta + 2hr\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)}{\begin{array}{c} -(2\gamma\beta + 2hr\|D\bar{f}(s(t))\| \sum_{i=2}^{N} \lambda_{max}(Z_i) + 2h\mu(t)\|D\bar{f}(s(t))\| \sum_{i=2}^{N} \lambda_{max}(Z_i) \\ +hr^2 \sum_{i=2}^{N} \lambda_{max}(Z_i) + h^2c^2r^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) \\ +h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T) \sum_{i=2}^{N} \lambda_{max}(\Sigma_i)\lambda_i^2\lambda_{max}^2(Z_i)) - \delta \end{array}},
$$

we have

$$
\dot{V} \leq -\delta\|w\|^2 + h\mu^2(t) \sum_{i=2}^{N} \lambda_{max}(Z_i)\lambda_i^2, \qquad (6.39)
$$

where Ξ is defined in (6.29). Thus, according to definition 4.1 and Lyapunov stability theory, bounded consensus is ultimately achieved.

Remark 6.2. The above two bounded consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical nodes to the cases of non-identical nodes. Because of the complexity of the consensus problems for non-identical nodes, we only obtain here sufficient conditions instead of sufficient and necessary condition. At the same time, the conditions obtained here are somewhat complicated and difficult to verify, but according to certain specific cases, we can construct an appropriate numerical simulation example to verify the effectiveness of the proposed results. Comparing the above two theorems, it can be seen that the boundary of the convergence set and the maximum size of time-varying delay can be evaluated respectively.

6.4 Global Controlled Bounded Consensus Criterion

6.4.1 Linear feedback pinning controller

To achieve the goal, we apply the feedback control strategy on a small fraction δ $(0 < \delta \leq 1)$ of the agents in system (6.1). Suppose that nodes $i_1, i_2, \cdots, i_l$ are selected to be under control, where $l = [\delta N]$ stands for the smaller but nearest integer to the real number δN. This controlled NMAS can be described as

$$\begin{cases} \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j\in\mathcal{N}_i} a_{i_kj}\Gamma x_j(t-\tau(t)) + u_{i_k}, 1 \leq k \leq l, \\ \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j\in\mathcal{N}_i} a_{i_kj}\Gamma x_j(t-\tau(t)), l+1 \leq k \leq N. \end{cases} \tag{6.40}$$

The local linear negative feedback control law is chosen as follows:

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k} - s), & 1 \le k \le l, \\ u_{i_k} = 0, & l+1 \le k \le N, \end{cases} \tag{6.41}$$

where the feedback gain $d_{i_k} > 0$.

Combine (6.40) and (6.41) and rearrange the order of the nodes in the network. Let the first l nodes be controlled, and $e_i = x_i - s,\ i = 1, 2, \cdots, N$. It's obvious that $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathbb{N}_i} a_{kj}\Gamma x_j(t-\tau(t)) = 0$ and $\sum_{i=1}^{N} e_i = 0$. Then by applying the Newton-Leibniz formula, error systems can be written as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathbb{N}_i} a_{ij}\Gamma e_j(t-\tau(t)) \\ \quad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad + f_i(s) - \bar{f}(s) - d_i e_i, \quad 1 \le i \le l, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathbb{N}_i} a_{ij}\Gamma e_j(t-\tau(t)) \\ \quad + \int_0^1 (Df_i + \tau e_i) - D\bar{f}(s))e_i d\tau - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad + f_i(s) - \bar{f}(s), \quad l+1 \le i \le N. \end{cases} \tag{6.42}$$

The following work will focus on simplifying the error systems (6.42) by means of a series of transformations using a procedure similar to [88].

Define the following matrix

$$D = diag(D_1, D_2, \cdots, D_N) \in R^{nN\times nN},$$

where $D_i = diag\{-d_i, -d_i, \cdots, -d_i\} \in R^{n\times n}$.

Let $e = (e_1^T, e_2^T, \cdots, e_N^T)^T$, then (6.42) becomes

$$\dot{e} = \bar{\Sigma}(t)e + cA \otimes \Gamma e(t - \tau(t)) + I(t)e - \frac{1}{N}H(t)e + F(t), \qquad (6.43)$$

where $\bar{\Sigma}(t) = I_N \otimes D\bar{f}(s) + D$, $H^T(t) = (H_1^T(t), \cdots, H_N^T(t))$, $H_i(t) = (\int_0^1 Df_1(s+\tau e_1)d\tau, \cdots, \int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t) = (f_1^T(s) - \bar{f}^T(s), \cdots, f_N^T(s) - \bar{f}^T(s))$ and $I(t) = diag\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau \cdots \int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\}$.
Since A is symmetric and irreducible, according to [88], there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \cdots, \Phi_N)$. This together with $w(t) = (\Phi^T \otimes I_n)e$ gives

$$\begin{aligned}\dot{w} = &(\Phi^T \otimes I_n)\bar{\Sigma}(t)(\Phi \otimes I_n)w + (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)w(t - \tau(t)) \\ &+ (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w + (\Phi^T \otimes I_n)F(t) - \frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)w.\end{aligned} \qquad (6.44)$$

Note that $H(t) = \sqrt{N}\sum_{k=1}^N (\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its k-th column equal to Φ_1 and the remaining elements are zero. Then we have $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^N (\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau(\Phi \otimes I_n)$, where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the remaining of its elements are zero.
Thus, a simple calculation gives $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \begin{pmatrix} \Upsilon_k & \mathtt{0} \end{pmatrix}^T \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k \in R^{1\times N}$ and $\mathtt{0} \in R^{(N-1)\times N}$. Therefore, $\dot{w} = \bar{\Sigma}(t)w + c\Lambda \otimes \Gamma w(t-\tau(t)) + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w - \begin{pmatrix} * & \mathtt{0} \end{pmatrix}^T w + (\Phi^T \otimes I_n)F(t)$. Since $w_1 \equiv 0$, we only need to consider $w_2, w_3, \cdots, w_N$. Rewriting in the compo-

nent form we have

$$\dot{w}_i = \Sigma_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau(t)) + (\Phi_i^T \otimes I_n)F(t)$$
$$+ (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w, \ i = 2, 3, \cdots, N, \tag{6.45}$$

where $\Sigma_i = \bar{D}f(s) + D_i$.

So far, we have transferred the consensus problem of system (6.1) to the stability problem of the $N-1$ of $n-$dimensional systems.

Theorem 6.3 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$a\|x\|^2 \leq x^T P_i(t)x + \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \leq b\|x\|^2,$$
$$\forall t \in R^+, \ x \in R^n, i = 2, 3, \cdots, N, \tag{6.46}$$

$$\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i$$
$$+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \zeta I \leq 0, \ i = 1, 2, \cdots, N, \tag{6.47}$$

$$\|I(t)\| \leq \gamma, \ i = 1, 2, \cdots, N. \tag{6.48}$$

Let

$$\mu(t) = \|F(t)\| \tag{6.49}$$

be bounded and

$$\beta = (\sum_{i=2}^{N} \|P_i(t)\|^2)^{\frac{1}{2}}, \tag{6.50}$$

if $\zeta > 2\gamma\beta$, then system (6.43) converges to the set

$$M = \{e | \|e\| \leq \frac{2b}{a} \frac{\beta \overline{lim}_{t\to\infty} \mu(t)}{\zeta - 2\gamma\beta - \delta}\}, \tag{6.51}$$

for any time-varying delay $\tau(t) > 0$, namely, $e(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^{N} x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, Furthermore, the NMAS (6.1) achieves bounded consensus for any fixed time delay $\tau(t) > 0$, $0 \leq \dot{\tau}(t) \leq 1$.

Proof. Choose the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N} V_i, \tag{6.52}$$

$$V_i = w_i^T P_i(t) w_i + \int_{t-\tau(t)}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha. \tag{6.53}$$

Differentiating (6.53) along the trajectory of (6.45) gives

$$\begin{aligned}
\dot{V}_i &= w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i)w_i \\
&+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \\
&+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) + 2w_i^T(c\lambda_i P_i(t)\Gamma)w_i(t-\tau(t)) \\
&- w_i^T(t-\tau(t))Q_i w_i(t-\tau(t)).
\end{aligned} \tag{6.54}$$

Applying the Young Inequality to the equality (6.54) results in

$$\begin{aligned}
\dot{V}_i &\leq w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i \\
&+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) \\
&+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w.
\end{aligned} \tag{6.55}$$

Condition (6.47) implies that the first term on the right hand side of (6.55) satisfies

$$\begin{aligned} w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i \\ + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i \leq -\zeta\|w_i\|^2. \end{aligned} \tag{6.56}$$

The second term on the right hand side of (6.55) satisfies

$$2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i\|. \tag{6.57}$$

Applying condition (6.48) we know the third term on the right hand side of (6.55) satisfies

$$2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \leq 2\gamma\|P_i(t)\|\|w_i\|\|w\|. \tag{6.58}$$

Since $V = \sum_{i=2}^N V_i$, we have

$$\begin{aligned} \dot{V} &= \sum_{i=2}^N \dot{V}_i \\ &= -\zeta\|w\|^2 + 2(\gamma\|w\| + \mu(t))\sum_{i=2}^N \|w_i\|\|P_i(t)\| \\ &\leq -\zeta\|w\|^2 + 2(\gamma\|w\| + \mu(t))\|w\|(\sum_{i=2}^N \|P_i(t)\|^2)^{\frac{1}{2}} \\ &= \|w\|((2\gamma\beta - \zeta)\|w\| + 2\beta\mu(t)). \end{aligned} \tag{6.59}$$

Thus, when

$$\|w\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \tag{6.60}$$

we have

$$\dot{V} \leq -\delta \|w\|^2. \tag{6.61}$$

Applying the result in [88] completes the proof.

6.4.2 Adaptive pinning controller

In this section, we will derive globally consensus criteria via direct adaptive pinning control method. Without loss of generality, we still assume that the first l agents are selected as pinned agents with the adaptive controllers:

$$\begin{cases} u_i = -d_i(x_i - s), & 1 \leq i \leq l, \\ \dot{d}_i = h_i e_i^T P_i(t) e_i, & \\ u_i = 0, & l+1 \leq i \leq N, \end{cases} \tag{6.62}$$

where constant $h_i > 0$ and positive definite matrix $P_i(t) \in R^{n\times n}$. Applying Newton-Leibniz formula, then the error NMAS can be rewritten as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathbb{N}_i} a_{ij}\Gamma e_j(t-\tau(t)) + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \qquad -\frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s) - d_i e_i, \quad 1 \leq i \leq l, \\ \dot{d}_i = h_i e_i^T P_i(t) e_i, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathbb{N}_i} a_{ij}\Gamma e_j(t-\tau(t)) + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \qquad -\frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s), \quad l+1 \leq i \leq N. \end{cases} \tag{6.63}$$

Repeating a similar procedure to the previous subsection, the controlled consensus problem of system (6.1) is equivalent to the stability problem of the following $N-1$

of n-dimensional systems.

$$\begin{cases} \dot{w}_i = D\bar{f}(s(t))w_i - d_i w_i + c\lambda_i \Gamma w_i(t-\tau(t)) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \qquad\qquad +(\Phi_i^T \otimes I_n)F(t), \qquad\qquad\qquad 2 \leq i \leq l, \\ \dot{d}_i = h_i w_i^T P_i(t) w_i, \\ \dot{w}_i = D\bar{f}(s)w_i + c\lambda_i \Gamma w_i(t-\tau(t)) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \qquad\qquad +(\Phi_i^T \otimes I_n)F(t), \qquad\qquad\qquad l+1 \leq i \leq N, \end{cases} \tag{6.64}$$

where w_i, w, Φ, Φ_i, $I(t)$ and $F(t)$ are the same as the previous subsection.

Theorem 6.4 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\bar{\zeta} > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$\begin{aligned} a\|x\|^2 \leq x_i^T P_i(t) x_i &+ \int_{t-\tau(t)}^{t} x_i^T(\alpha) Q_i x_i(\alpha) d\alpha \\ &+ \frac{(d_i - d)^2}{h_i} \leq b\|x\|^2, \forall t \in R^+,\ x \in R^n, i = 2, 3, \cdots, N, \end{aligned} \tag{6.65}$$

$$\begin{aligned} \dot{P}_i(t) + &P_i(t) D\bar{f}(s) + (D\bar{f}(s))^T P_i(t) + Q_i - 2dP_i(t) \\ &+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \bar{\zeta} I \leq 0,\ i = 1, 2, \cdots, N, \end{aligned} \tag{6.66}$$

(6.48) and $\bar{\zeta} > 2\gamma\beta$ are satisfied, then the system (6.43) converges to the set (6.51) for any time-varying delay $\tau(t) > 0$, where $\mu(t)$ and β are the same as in (6.49) and (6.50) respectively, $\bar{\delta} > 0$ is any constant satisfying $\bar{\delta} < \bar{\zeta} - 2\gamma\beta$, and then the NMAS (6.1) achieves bounded consensus for any fixed time delay $\tau(t) > 0$, $0 \leq \dot{\tau}(t) \leq 1$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N} V_i + \sum_{i=2}^{l} \frac{(d_i - d)^2}{h_i}, \tag{6.67}$$

where

$$\begin{cases} V_i = w_i^T P_i(t) w_i + \int_{t-\tau(t)}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha + \frac{(d_i-d)^2}{h_i},\ 2 \le i \le l, \\ V_i = w_i^T P_i(t) w_i + \int_{t-\tau(t)}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha,\ l+1 \le i \le N, \end{cases} \tag{6.68}$$

where d is a positive constant to be determined.

Differentiating (6.68) along the trajectory of (6.64) gives

$$\begin{aligned} \dot{V}_i = {} & w_i^T(\dot{P}_i(t) + P_i(t) D\bar{f}(s) + (D\bar{f}(s))^T P_i(t) + Q_i - 2dP_i(t)) w_i \\ & + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n) I(t)(\Phi_i \otimes I_n) w + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n) F(t) \\ & + 2w_i^T(c\lambda_i P_i(t)\Gamma) w_i(t-\tau(t)) - w_i^T(t-\tau(t)) Q_i w_i(t-\tau(t)). \end{aligned} \tag{6.69}$$

The remaining part of the proof is similar to that of Theorem 6.1, so is therefore omitted here. This completes the proof.

6.5 Examples

Example 6.1

In this subsection, we will construct an example to demonstrate the consensus analysis results proposed above.

The agent dynamics can be chosen as follows

$$\dot{x}_i(t) = B_i x_i(t) + g(x_i(t)), \quad i = 1, 2, \cdots, 11, \tag{6.70}$$

where

$$\begin{cases} B_i = \begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix}, & i=1,2,\cdots,6, \\ B_i = \begin{pmatrix} -10-0.1\times(i-6) & 10+0.1\times(i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15+0.1\times(i-6) & 0 \end{pmatrix}, & i=7,8,\cdots,11, \end{cases}$$

and

$$g(x_i(t)) = (-9.5sin(\frac{\pi x_{i1}(t)}{3.2}+\pi)\ 0\ 0)^T, \quad i=1,2,\cdots,11.$$

The communication coupling matrix A and the inner coupling matrix are

$$A = \begin{pmatrix} -8 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -8 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -6 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & -7 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -5 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -10 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -6 \end{pmatrix}, \Gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

respectively.

Given the initial values of 11 agents as $(10\ 5\ -10)^T$, $(12\ 6\ -12)^T$, $(14\ 7\ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$,

$(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$ and $(-10\ 15\ 10)^T$ respectively. We may verify the conditions of Theorem 6.1 readily. This demonstrates the consensus of the NMAS is achieved for the time varying delay $\tau(t) > 0$. Simulation results are depicted in Fig 6.1 to Fig 6.5 for $\tau(t) = 1 + 0.5sin^2(t))$ and $c = 1$.

The simulation curves in Fig 6.1 show that the states of all agents are ultimately bounded stable. The average state trajectory $s(t)$ is chosen as the desired moving trajectory and is depicted in Fig 6.2. Fig 6.3 to Fig 6.5 demonstrate that the state errors between each agent's states and the desired state trajectory respectively, and the deviation systems are also ultimately bounded stable. These simulation curves show that all agents eventually move with the desired state trajectory in the sense of boundedness.

Example 6.2 To demonstrate the controlled consensus results obtained above, we construct anther NMAS consisting of 12 agents described as follows

$$\dot{x}_i(t) = f_i(x_i(t)) + c \sum_{j \in \mathcal{N}_i} a_{ij} \Gamma x_j(t - \tau(t)), \tag{6.71}$$

where $f_i(x_i(t)) = B_i x_i(t) + g(x_i(t))$, $B_i(i = 1, 2, \cdots, 6)$ and $B_i(i = 7, 8, \cdots, 12)$ are chosen as follows:

$$\begin{pmatrix} -10 + 0.1 \times (i-1) & 10 - 0.1 \times (i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15 - 0.1 \times (i-1) & 0 \end{pmatrix},$$

$$\begin{pmatrix} -10 - 0.1 \times (i-6) & 10 + 0.1 \times (i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15 + 0.1 \times (i-6) & 0 \end{pmatrix},$$

and

$$g(x_i(t)) = (-9.5sin(\frac{\pi x_{i1}(t)}{3.2} + \pi)\ 0\ 0)^T, \quad i = 1, 2, \cdots, 12.$$

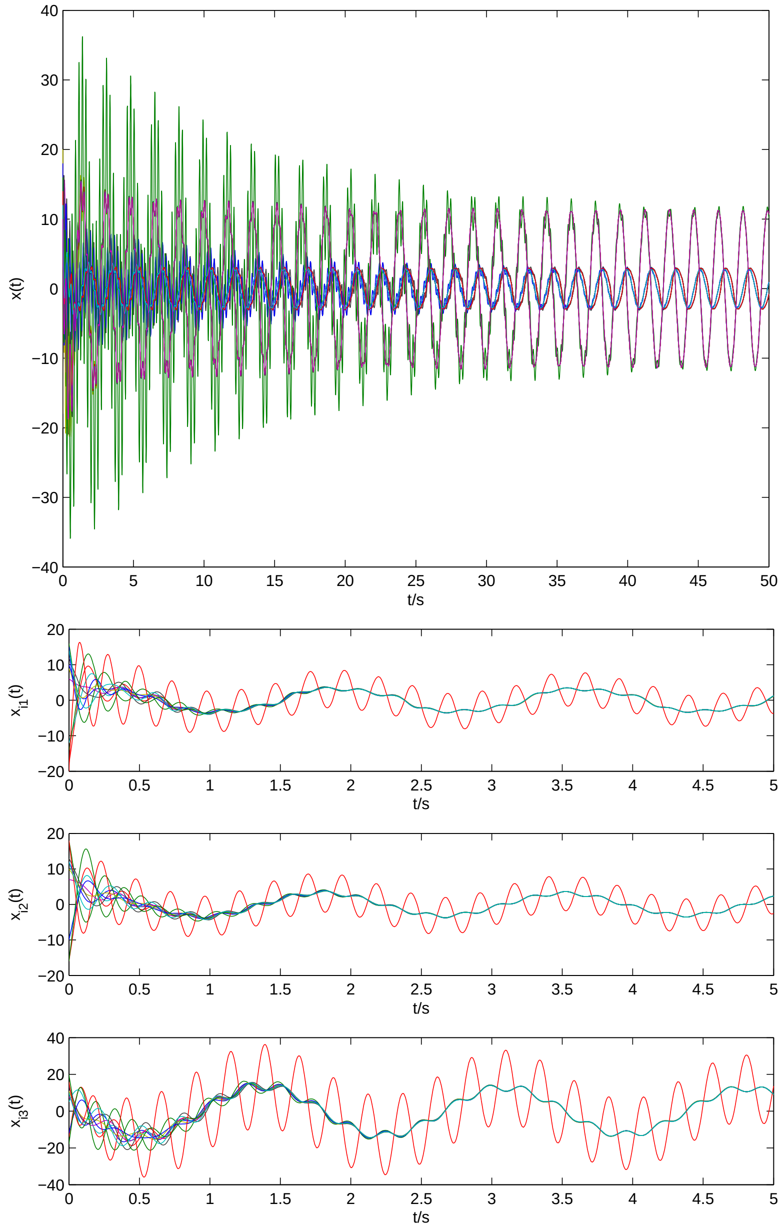

Fig 6.1 The dynamics of all agents.

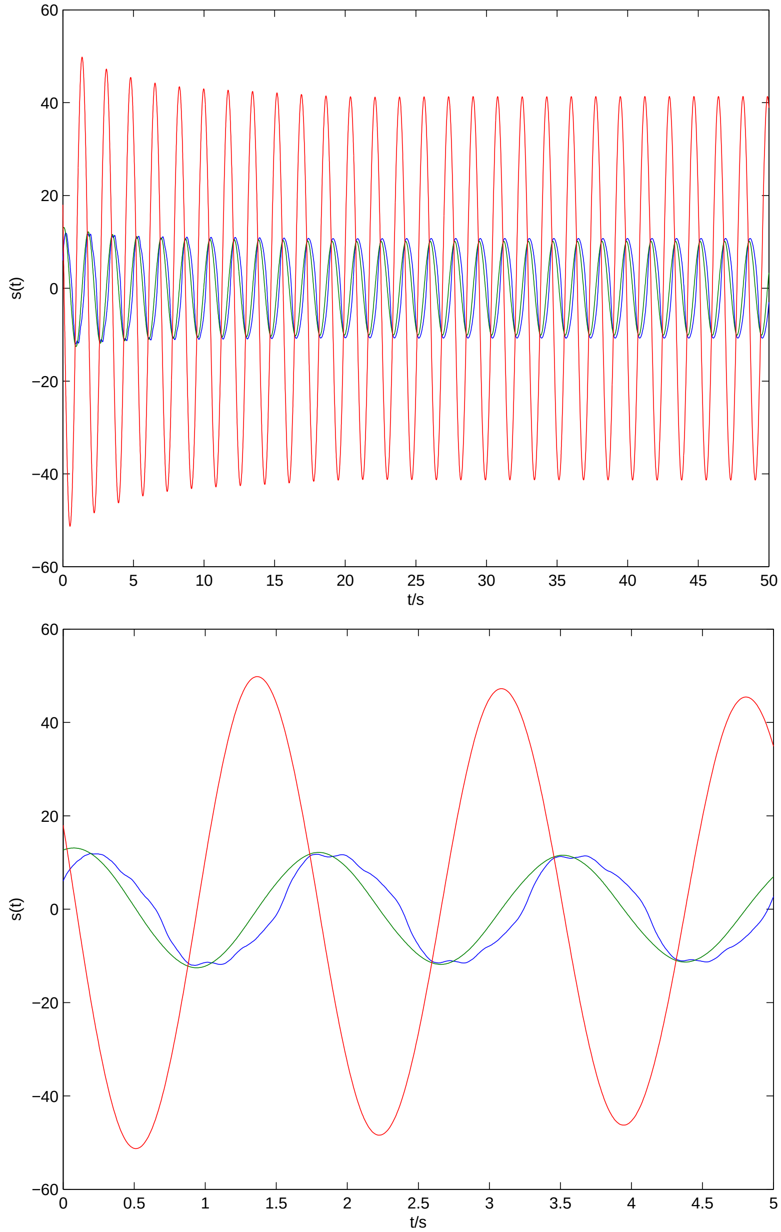

Fig 6.2 The average state trajectory $s(t)$.

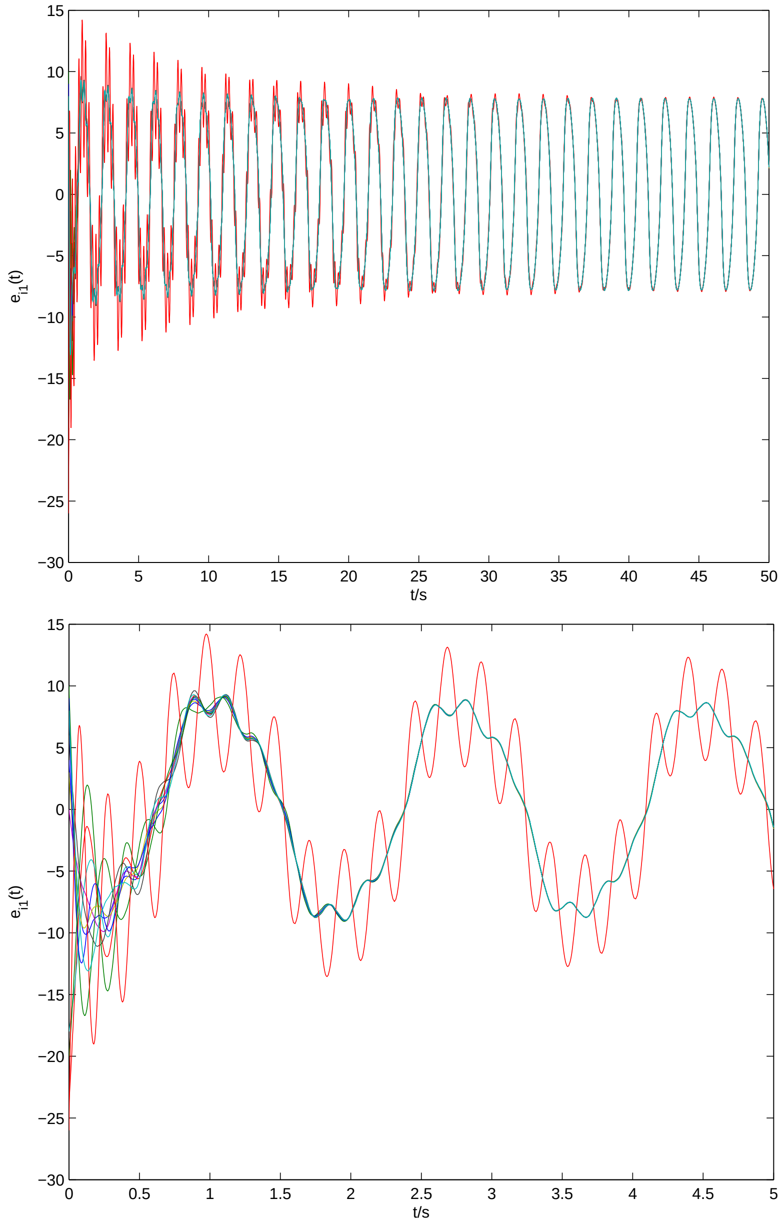

Fig 6.3 The consensus error dynamics for the first dynamic of each agent.

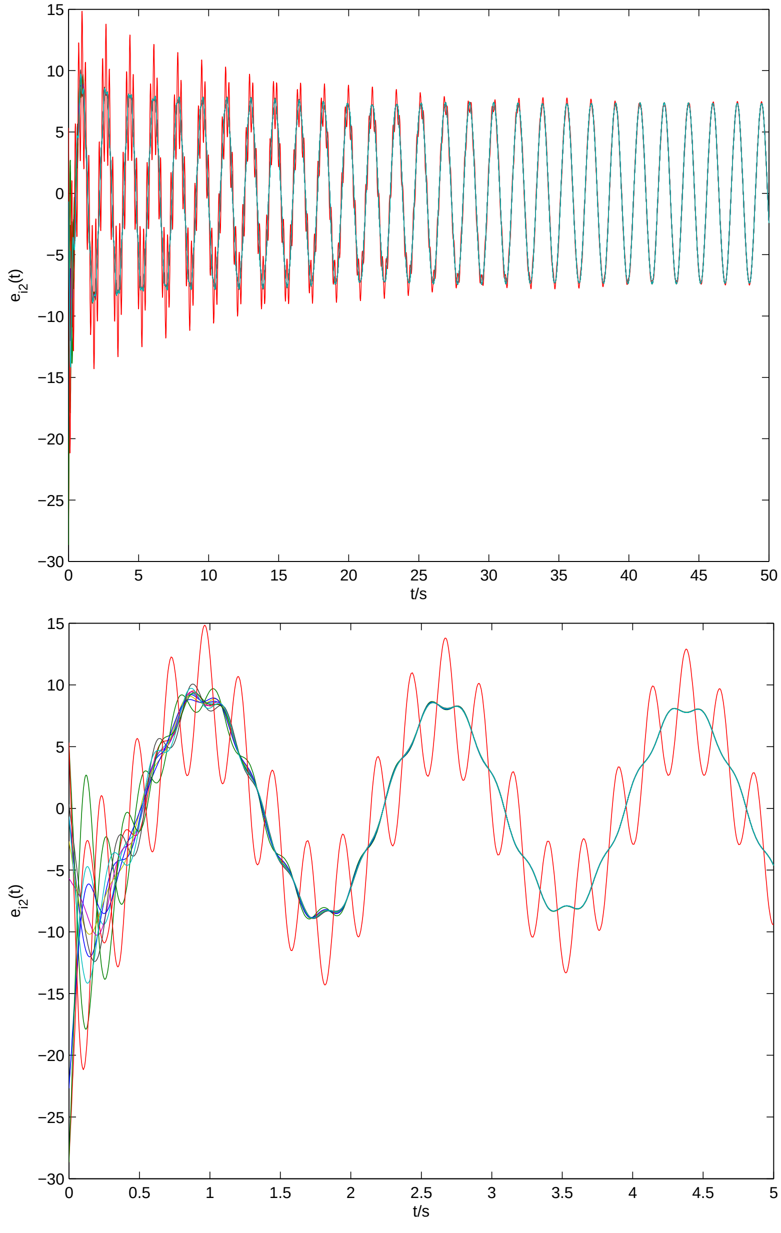

Fig 6.4 The consensus error dynamics for the second dynamic of each agent.

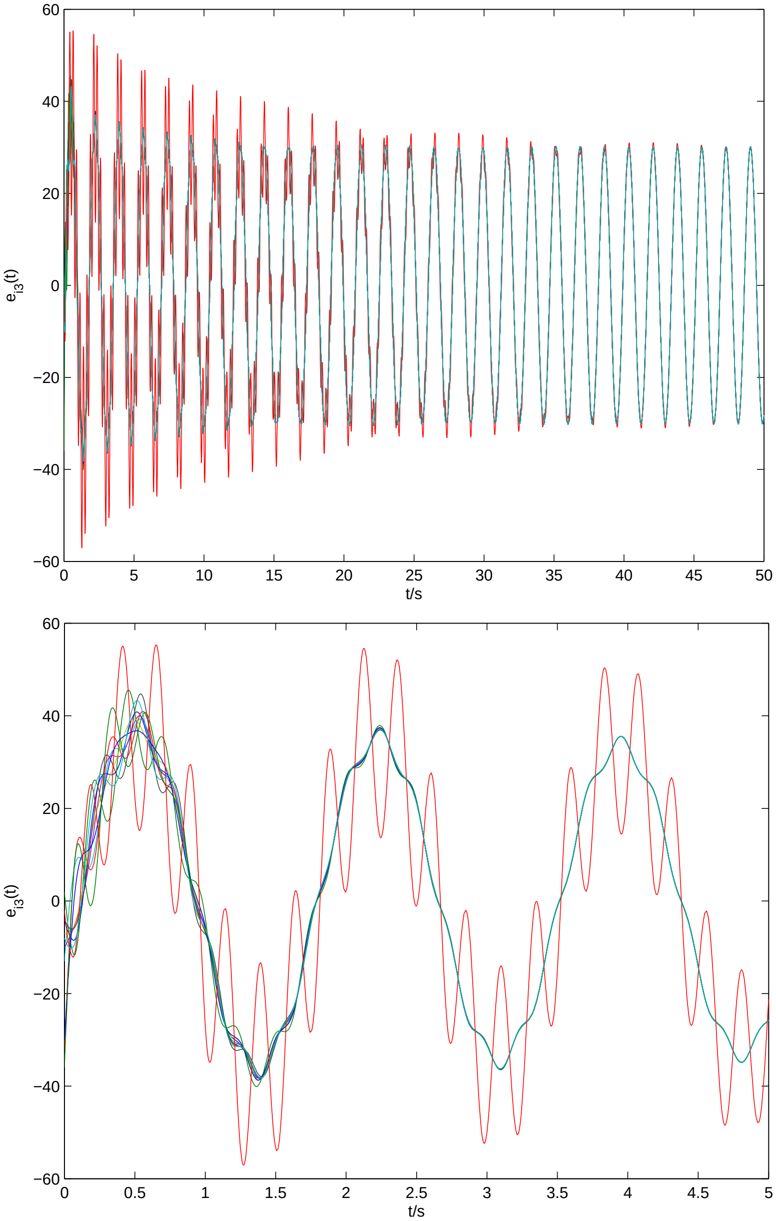

Fig 6.5 The consensus error dynamics for the third dynamic of each agent.

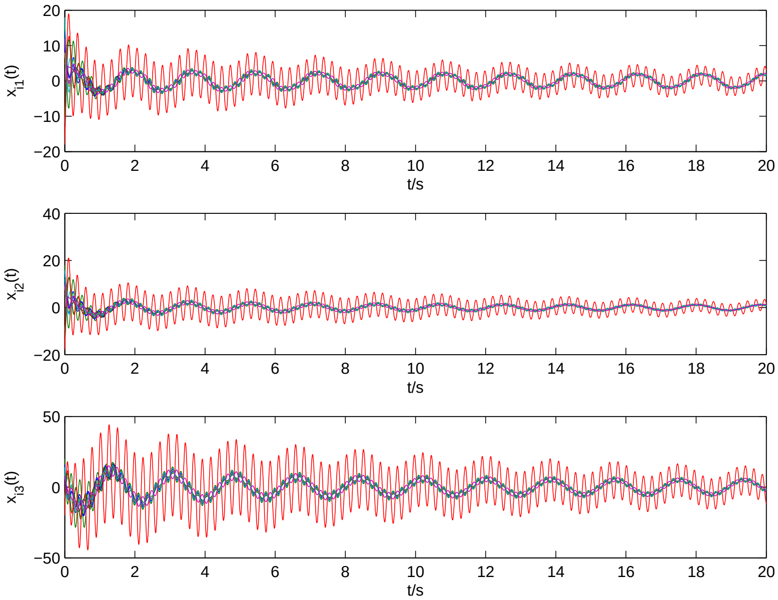

Fig 6.6 All agent dynamics under pinning control.

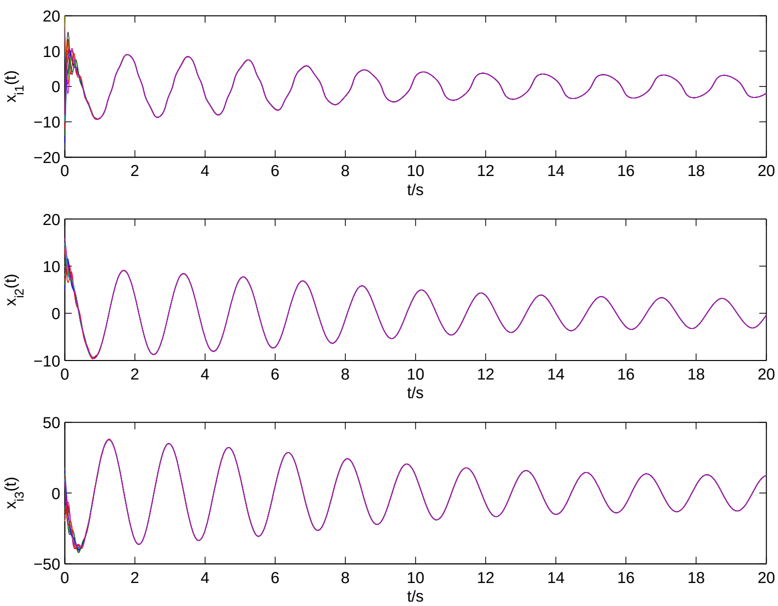

Fig 6.7 All agent dynamics under adaptive pinning control.

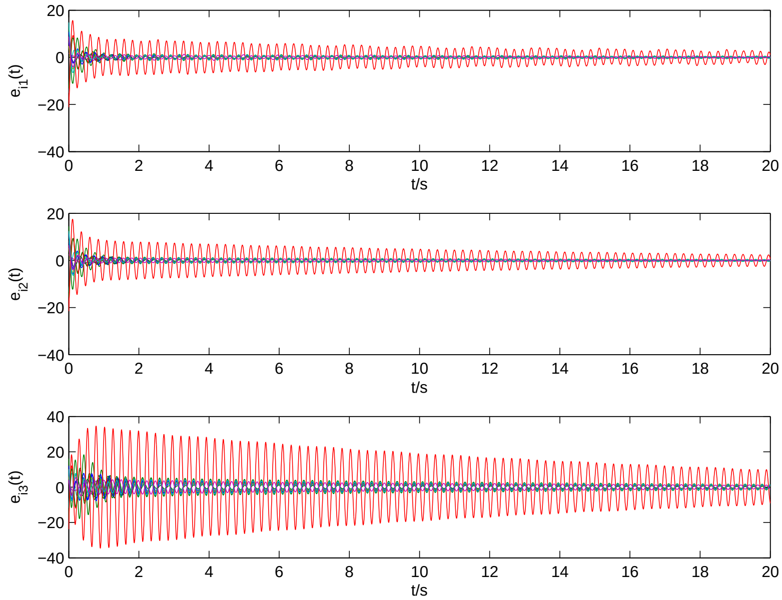

Fig 6.8 All agent error dynamics under pinning control.

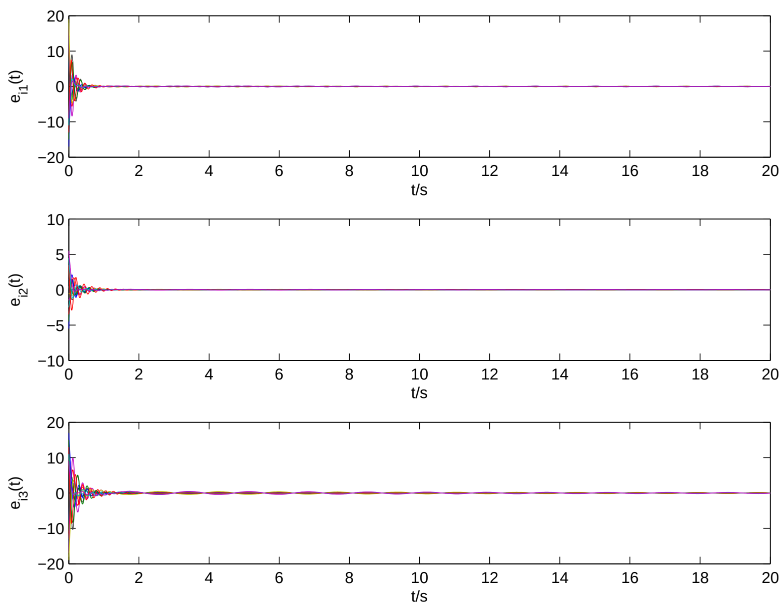

Fig 6.9 All agent error dynamics under adaptive pinning control.

The coupling matrix $C = (C_1^T C_2^T \cdots C_{12}^T)$, $C_1 = (-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0)$, $C_2 = (1\ -8\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0)$, $C_3 = (1\ 1\ -7\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1)$, $C_4 = (0\ 1\ 1\ -6\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1)$, $C_5 = (1\ 1\ 0\ 0\ -6\ 0\ 1\ 1\ 1\ 1\ 0\ 0)$, $C_6 = (1\ 0\ 0\ 1\ 0\ -5\ 1\ 0\ 1\ 1\ 0)$, $C_7 = (1\ 1\ 0\ 1\ 1\ 1\ -7\ 1\ 0\ 1\ 0)$, $C_8 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ -6\ 0\ 1\ 1\ 1)$, $C_9 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ -7\ 1\ 1\ 1)$, $C_{10} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ -10\ 1)$, $C_{11} = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ -7\ 1)$, $C_{12} = (0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ -5)$. $\Gamma = diag\{2, 2, 2\}$, respectively, where the matrix A is produced by means of the Scale-Free network program.

Design the following controllers

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k}(t) - s(t)), & i_k = 1, 2 \texttt{ and } 10, \\ u_{i_k} = 0, & \texttt{else}, \end{cases}$$

with $d_1 = 0.5$, $d_2 = 0.5$, $d_{10} = 0.5$ and

$$\begin{cases} u_{i_k} = -d_{i_k}(t)(x_{i_k}(t) - s(t)), & i_k = 1, 2 \texttt{ and } 10, \\ \dot{d}_{i_k}(t) = h_{i_k} e_{i_k}^T P_{i_k}(t) e_{i_k}, & \\ u_{i_k} = 0, & \texttt{else}, \end{cases}$$

with $h_1 = 0.1$, $h_2 = 0.2$, $h_{10} = 0.3$, $s(t)$ can then be evaluated by simulation.

Given the initial values of 12 agents as $(10\ 5\ -10)^T$, $(12\ 6\ -12)^T$, $(14\ 7\ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$, $(-10\ 15\ 10)^T$, $(-8\ 16\ 8)^T$ respectively and $P_{i_k}(t) = I_3$, $d_1(0) = 1$, $d_2(0) = 1$, $d_{10}(0) = 1$ and $\tau(t) = \frac{\pi}{2} + arctan(t)$. The conditions of Theorem 6.3 and Theorem 6.4 are satisfied readily. Bounded consensus of the NMAS is achieved for any time varying delay satisfying $0 < \tau \leq \frac{\pi}{2} + arctan(t)$. Simulation results are depicted in Fig 6.6 to Fig 6.9 for $\tau(t) = \frac{\pi}{2} + arctan(t)$ and $c = 1$.

6.6 Conclusions

In this chapter, we've investigated the consensus problems of NMAS with different agent dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on a series of transformations and Lyapunov stability theorem, the proposed results have also been extended to the controlled consensus problems based on pining control and adaptive pining control scheme. Many related results for the case of identical agent dynamics have been viewed as the special cases of the proposed results. However, it should be noted that the conditions are still restrictive and the time-varying delay is chosen as special case.

Chapter 7

Global Bounded Consensus of NMAS with Different Agents and Multiple Time Delays

---

This chapter investigates the global bounded consensus problem of NMAS consisting of nonlinear, non-identical node dynamics with communication time-delay topology. We derive globally bounded controlled consensus conditions for both delay-independent and delay-dependent conditions based on the Lypunov-Krasovskii functional method. The proposed consensus criteria ensures that all agents eventually move along the desired trajectory in the sense of boundedness. Meanwhile, the bounded consensus criteria can be viewed as an extension of the case of identical agent dynamics to the case of of non-identical agent dynamics. We finally demonstrate the effectiveness of the theoretical results by means of a numerical simulation.

7.1 Introduction

The consensus problem is to design distributed control strategies based on local information that enables all agents to reach some kind of agreements on certain quantities of interest. The topic has been studied across many fields of science and engineering. The consensus analysis of a NMAS consisting of non-identical agent dynamics is much more complicated than the identical case and few results have been reported to date. This chapter will focus on the global consensus problems of this kind of NMAS. Globally bounded consensus criterion and controlled bounded consensus criterion based on the Lyapunov-Krasovskii functional method and pining control scheme are obtained.

The rest of this chapter is organized as follows. A continuous-time NMAS model with non-identical agent dynamics and communication time delays is presented in section 7.2. The main results including delay-independent and delay-dependent bounded consensus criterion are derived in section 7.3 and its corresponding controlled bounded consensus problems are discussed in section 7.4. Numerical simulation examples are given in section 7.5 to verify the effectiveness of the proposed results, followed by conclusions in section 7.6.

7.2 Problem Description

Consider a NMAS consisting of N non-identical agents with communication delay:

$$\dot{x}_i(t) = f_i(x_i(t)) + u_i,\ i = 1, 2, \cdots, N, \tag{7.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i(t)) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $u_i = c\sum_{j\in\mathbb{N}_i} a_{ij}\Gamma(x_{j1}(t -$

$\tau_{j1}), x_{j2}(t-\tau_{j2}), \cdots, x_{jn}(t-\tau_{jn}))^T$, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix, and if $\gamma_{ij} \neq 0$, then it means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N\times N}$, which is symmetric and irreducible, representing the communication topology relation of the NMAS, is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j\neq i} a_{ij}$. Time delay vector $(\tau_{j1}, \tau_{j2}, \cdots, \tau_{jn})$ reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

Remark 7.1 The consensus problem of NMAS is usually viewed as the synchronization of coupled nonlinear oscillators and there is no significant differences between the NMAS and the complex dynamical networks. The synchronization problem of the complex dynamical network usually emphasizes their nonlinear node dynamics, thus only sufficient conditions can be given for verifying the synchronization. However, in the context of NMAS, researches are mainly focusing on designing various distributed strategies which can guarantee the NMAS achieve agreements. In many cases, the agent dynamics are usually restricted to be single or double integrators or high-order linear systems, and the most proposed distributed consensus protocols are usually based on the relative states between neighboring agents, but the fact is that many necessary and sufficient conditions can be given.

The average dynamics of all agents is defined by the vector field

$$\bar{f}(x(t)) = \frac{1}{N}\sum_{k=1}^{N} f_k(x(t)). \tag{7.2}$$

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t). \tag{7.3}$$

The consensus problem formulation in the present paper is still the same to the previous chapter. That is to say, the consensus problem is solvable if the states of all agents satisfy certain boundedness of $x_i(t) - x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. This better reflects reality when it is impossible for NMAS (7.1) to achieve exact consensus. To address this case we will focus on making the states of all agents converge to a bounded set.

We now introduce some notations and definitions.

Let $\overline{x_j(t-\tau)} = (x_{j1}(t-\tau_1), x_{j2}(t-\tau_2), \cdots, x_{jn}(t-\tau_n))^T$, $\overline{x_j(t-\tau_j)} = (x_{j1}(t-\tau_{j1}), x_{j2}(t-\tau_{j2}), \cdots, x_{jn}(t-\tau_{jn}))^T$.

Let $\mu_2(A)$ is half the maximum eigenvalue of $\bar{A}^T + A$.

7.3 Globally Bounded Consensus Analysis

Define the error vector

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \cdots, N. \tag{7.4}$$

Obviously, $\sum_{i=i}^{N} e_i = 0$ and $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i}^{N} a_{kj}\Gamma\overline{x_j(t-\tau_j)} = 0$, then the NMAS (7.1) can be rewritten in terms of e_i as

$$\dot{e}_i(t) = f_i(s(t)+e_i(t)) - \frac{1}{N}\sum_{k=1}^{N} f_k(s(t)+e_k(t)) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma\overline{e_j(t-\tau_j)}, \tag{7.5}$$

where $\overline{e_j(t-\tau_j)} = (e_{j1}(t-\tau_{j1}), e_{j2}(t-\tau_{j2}), \cdots, e_{jn}(t-\tau_{jn}))^T$.

The following work will focus on simplifying the error NMAS (7.5) by means of a series of transformations and the procedure is similar to [88, 89].

According to Newton-Leibniz formula, the error NMAS (7.5) can be written further as

$$\begin{aligned}\dot{e}_i(t) =& D\bar{f}(s(t))e_i(t) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma\overline{e_j(t-\tau_j)}\\ &+\int_0^1 (Df_i(s(t)+\tau e_i(t)) - D\bar{f}(s(t)))e_i(t)d\tau\\ &-\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s(t)+\tau e_k(t))e_k(t)d\tau + f_i(s(t)) - \bar{f}(s(t)),\end{aligned}\tag{7.6}$$

where $D\bar{f}(s(t))$ is the Jacobian Matrix of $\bar{f}(x(t))$ at $s(t)$, and $\bar{f}(x(t))$ is defined by (refaveragedynamicc7).

If we consider the linearized NMAS of (7.5), we have

$$\begin{aligned}\dot{e}_i(t) = & D\bar{f}(s(t))e_i(t) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma\overline{e_j(t-\tau_j)}\\ &+(Df_i(s(t)) - D\bar{f}(s(t)))e_i(t) - \frac{1}{N}\sum_{k=1}^{N} Df_k(s(t))e_k(t) + f_i(s(t)) - \bar{f}(s(t)).\end{aligned}\tag{7.7}$$

Let $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$, then (7.6) becomes

$$\dot{e}(t) = I_N\otimes D\bar{f}(s)e(t) + cA\otimes\Gamma\overline{e(t-\tau_j)} + I(t)e(t) - \frac{1}{N}H(t)e(t) + F(t), \tag{7.8}$$

where $\bar{\Sigma}(t) = I_N\otimes D\bar{f}(s) + D$, $H^T(t) = (H_1^T(t), \cdots, H_N^T(t))$, $H_i(t) = (\int_0^1 Df_1(s+\tau e_1)d\tau, \cdots, \int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t) = (f_1^T(s) - \bar{f}^T(s), \cdots, f_N^T(s) - \bar{f}^T(s))$ and $I(t) = diag\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau \cdots \int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\}$.

Since the graph $G = (\mathcal{V}, \mathcal{A})$ is a strongly connected graph and A is symmetric and irreducible, there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \cdots, \Phi_N)$, such that the adjacency matrix A satisfies $\Phi^T A\Phi = \Lambda = diag\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$, where Φ_i is the i-th column of Φ with $\Phi_1 = (\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}})^T$ and $0 = \lambda_1 >$

$\lambda_2 \geq \cdots \geq \lambda_N$ are the eigenvalues of A. Let $\omega(t) = (\Phi^T \otimes I_n)e(t)$, then

$$\begin{aligned}\dot{\omega}(t) &= (\Phi^T \otimes I_n)\dot{e}(t) = (\Phi^T \otimes I_n)(I_N \otimes D\bar{f}(s))(\Phi \otimes I_n)\omega(t) \\ &\quad + (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)\overline{\omega(t-\tau_j)} + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\ &\quad - \frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)\omega(t) + (\Phi^T \otimes I_n)F(t). \end{aligned} \tag{7.9}$$

Note that $H(t) = \sqrt{N}\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its k-th column equal to Φ_1 and the remaining elements are zero, then we have $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau(\Phi \otimes I_n)$, where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the remaining of its elements are zero. Thus, $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\begin{pmatrix}\Upsilon_k \\ \mathsf{0}\end{pmatrix} \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k \in R^{1\times N}$ and $\mathsf{0} \in R^{(N-1)\times N}$. Therefore, $\dot{\omega}(t) = I_N \otimes D\bar{f}(s)\omega(t) + c\Lambda \otimes \Gamma\overline{\omega(t-\tau_j)} + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) - \begin{pmatrix} * \\ \mathsf{0}\end{pmatrix}\omega(t) + (\Phi^T \otimes I_n)F(t)$.. Since $w_1 \equiv 0$, we only need to consider $w_2, w_3, \cdots, w_N$. Rewriting in the component form we have

$$\begin{aligned}\dot{\omega}_i(t) &= D\bar{f}(s(t)\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)} + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\ &\quad + (\Phi_i^T \otimes I_n)F(t),\ i = 2, 3, \cdots, N. \end{aligned} \tag{7.10}$$

In the following, a delay-independent global consensus criteria is derived for the NMAS (7.1).

Theorem 7.1 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$ and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$\begin{aligned} a\|x(t)\|^2 \leq & x^T(t)P_i(t)x(t) \leq b\|x(t)\|^2, \\ & \forall t \in R^+,\ x \in R^n, i = 2, 3, \cdots, N, \end{aligned} \tag{7.11}$$

$$\mu_2(\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma) + \zeta I \leq 0,\ i = 1, 2, \cdots, N, \tag{7.12}$$

$$\|I(t)\| \leq \gamma,\ i = 1, 2, \cdots, N. \tag{7.13}$$

and exist an i and a j such that

$$\begin{pmatrix} -\lambda_j(\Gamma + \Gamma^T) & \lambda_i\Gamma \\ \lambda_i\Gamma^T & 0 \end{pmatrix} \leq 0, \tag{7.14}$$

for all $t \geq t_0$, where $i = 2, 3, \cdots, n$, $j = 1, 2, \cdots, n$.
Let

$$\mu(t) = \|F(t)\| \tag{7.15}$$

be bounded and

$$\beta = (\sum_{i=2}^{N} \|P_i(t)\|^2)^{\frac{1}{2}}, \tag{7.16}$$

if $\zeta > 2\gamma\beta$, then the system (7.46) converges to the set

$$M = \{e(t) | \|e(t)\| \leq \frac{2b}{a} \frac{\beta \overline{lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\}, \tag{7.17}$$

namely, $e(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^{N} x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, and then the NMAS (7.1) achieves global bounded consensus for any fixed time delays $\tau_{kl} > 0 (k, l = 1, 2, \cdots, n)$.

Proof. Choose the following Lyapunov function as

$$V(w_i(t), t) = \sum_{i=2}^{N} V_i(w_i(t), t), \tag{7.18}$$

$$V_i(w_i(t), t) = w_i^T(t) P_i(t) w_i(t), \quad i = 2, 3, \cdots, N. \tag{7.19}$$

Differentiating (7.19) along the trajectory of (7.48) gives

$$\begin{aligned}
&\dot{V}_i(w_i(t), t) = w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t))w_i(t) \\
&+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \\
&+ 2w_i^T(t)(c\lambda_i P_i(t)\Gamma)\overline{w_i(t-\tau_j)} \\
&= w_i^T(t)((\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma) \\
&+ (\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma)^T)w_i(t) \\
&+ c\left(\frac{w_i(t)}{w_i(t-\tau_j)}\right)^T \Theta \left(\frac{w_i(t)}{w_i(t-\tau_j)}\right) \\
&+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t).
\end{aligned} \tag{7.20}$$

where $\Theta = \begin{pmatrix} -\lambda_j(\Gamma + \Gamma^T) & \lambda_i\Gamma \\ \lambda_i\Gamma^T & 0 \end{pmatrix}$.

Condition (7.12) implies that the first term on the right hand side of (7.20) satisfies

$$\begin{aligned}
&w_i^T(t)((\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma) + (\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma)^T)w_i(t) \\
&\quad \le -\zeta\|w_i(t)\|^2.
\end{aligned} \tag{7.21}$$

Applying condition (7.13) we know the second term on the right hand side of

(7.20) satisfies

$$2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \leq 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|. \tag{7.22}$$

The third term on the right hand side of (7.20) satisfies

$$2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i(t)\|. \tag{7.23}$$

Since $V(w(t),t) = \sum_{i=2}^{N} V_i(w_i(t),t)$, we have

$$\begin{aligned}
\dot{V}(w(t),t) &= \sum_{i=2}^{N} \dot{V}_i(w_i(t),t) \\
&\leq \sum_{i=2}^{N}(-\zeta\|w_i(t)\|^2) + 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\| + 2\mu(t)\|P_i(t)\|\|w_i(t)\| \\
&= -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\| + \mu(t))\sum_{i=2}^{N}\|w_i(t)\|\|P_i(t)\| \\
&\leq -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\| + \mu(t))\|w(t)\|(\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}} \\
&= \|w(t)\|((2\gamma\beta - \zeta)\|w(t)\| + 2\beta\mu(t)).
\end{aligned} \tag{7.24}$$

Thus, when

$$\|w(t)\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \tag{7.25}$$

we have

$$\dot{V}(w(t),t) \leq -\delta\|w(t)\|^2. \tag{7.26}$$

Applying Lemma 4.1 completes the proof.

Remark 7.2. The above result is a delay-independent globally consensus criterion and the ultimate convergence bound is evaluated by means of (7.17). Theorem 7.1 guarantees that all agents move along the desired trajectory described by $s(t)$ in terms of certain boundedness, i.e., the consensus achieved here is just approximate instead of exact; in fact, to achieve exact consensus is impossible for such a case.

Remark 7.3 We have an asymptotic consensus criterion in the classical sense when $\overline{lim}_{t\to\infty}\mu(t) = 0$. In particular, we have $\mu(t) \equiv 0$ when all agents have the same dynamics, i.e., $f_i(x_i(t)) = f(x(t))$. In such a case, applying theorem 3.1 to the linearized network (7.7), which is equivalent to taking $\gamma = 0$ in (7.13), immediately achieves the universal consensus criteria existing in many literatures. Therefore, Theorem 7.1 covers the existing criteria of networks with identical agent dynamics as a special case.

Next, we will provide delay-dependent bounded consensus criterion for the proposed problem.

Theorem 7.2 Suppose that (7.11) and (7.13) in Theorem 3.1 are satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Pi_i > 0$, $Z_i > 0$, $\Sigma_i > 0$, X_i and Y_i of appropriate dimensions such that

$$\Xi = \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{pmatrix} < 0, \tag{7.27}$$

where

$$\begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{7.28}$$

for $i = 2, 3, \cdots, N$, $\Xi_{11} = \dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + nhX_i + nQ_i + 2n(Y_i - \frac{c\lambda_i}{n}P_i(t)\Gamma) + nhD\bar{f}(s(t))^T Z_i D\bar{f}(s(t))$, $\Xi_{12} = \frac{c\lambda_i}{n}P_i(t)\Gamma - Y_i + c\lambda_i nhD\bar{f}(s(t))^T Z_i\Gamma$, $\Xi_{22} = \Pi_i^{-1} + \Sigma_i^{-1} - Q_i + c^2\lambda_i^2 nh\Gamma^T Z_i\Gamma$ and then the NMAS

(7.1) achieves global bounded consensus for any fixed time delays $\tau_{kl} \in [0, h] > 0 (k, l = 1, 2, \cdots, n)$ for some $h < \infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V(w_i(t), t) = \sum_{i=2}^{N} \sum_{k=1}^{3} V_k(w_i(t), t), \tag{7.29}$$

where

$$\begin{aligned} V_1(w_i(t), t) &= w_i^T(t) P_i(t) w_i(t), \\ V_2(w_i(t), t) &= \sum_{l=1}^{n} \int_{-\tau_{kl}}^{0} \int_{t+\beta}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha d\beta, \\ V_3(w_i(t), t) &= \sum_{l=1}^{n} \int_{t-\tau_{kl}}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha. \end{aligned}$$

The i-th $(i = 2, 3, \cdots, N)$ equation in system (7.48) can be written as

$$\begin{aligned} \dot{\omega}_i(t) = (D\bar{f}(s(t)) + c\lambda_i \Gamma)\omega_i(t) - \frac{c\lambda_i}{n} \Gamma \sum_{l=1}^{n} \int_{t-\tau_{kl}}^{t} \dot{w}_i(\alpha) d\alpha \\ + (\Phi_i^T \otimes I_n) I(t) (\Phi \otimes I_n) \omega(t) + (\Phi_i^T \otimes I_n) F(t), \quad i = 2, 3, \cdots, N, \end{aligned} \tag{7.30}$$

and, thus, the derivative of $V_1(w_i(t), t)$ satisfies

$$\begin{aligned} \dot{V}_1(w_i(t), t) &= w_i^T(t)(\dot{P}_i(t) + P_i(t)(D\bar{f}(s(t)) + c\lambda_i \Gamma) \\ &+ (D\bar{f}(s(t)) + c\lambda_i \Gamma)^T P_i(t)) w_i(t) - \frac{2c\lambda_i}{n} w_i^T(t) P_i(t) \Gamma \sum_{l=1}^{n} \int_{t-\tau_{kl}}^{t} \dot{w}_i(\alpha) d\alpha \\ &+ 2w_i^T(t) P_i(t)(\Phi_i^T \otimes I_n) I(t) (\Phi_i \otimes I_n) w(t) + 2w_i^T(t) P_i(t)(\Phi_i^T \otimes I_n) F(t). \end{aligned} \tag{7.31}$$

Defining $a(.)$, $b(.)$ and M as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$ and $M = \frac{c\lambda_i}{n} P_i(t)\Gamma$ for all $\alpha \in [t-\tau_{kl}, t]$ and then applying Lemma 4.2 results in

$$
\begin{aligned}
\dot{V}_1(w_i(t),t) &\leq w_i^T(t)(\dot{P}_i(t) + P_i(t)(D\bar{f}(s(t)) + c\lambda_i\Gamma) \\
&\quad + (D\bar{f}(s(t)) + c\lambda_i\Gamma)^T P_i(t))w_i(t) + nhw_i^T(t)X_i w_i(t) \\
&\quad + 2w_i^T(t)(Y_i - \frac{c\lambda_i}{n}P_i(t)\Gamma)\sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i(\alpha)d\alpha + \sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha \\
&\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t)
\end{aligned}
$$

$\dot{V}_1(w_i(t),t)$ can be further enlarged as

$$
\begin{aligned}
&\dot{V}_1(w_i(t),t) \leq w_i^T(t)[\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) \\
&+ nhX_i + 2n(Y_i - \frac{c\lambda_i}{n}P_i(t)\Gamma)]w_i(t) + 2w_i^T(t)(\frac{c\lambda_i}{n}P_i(t)\Gamma - Y_i)\sum_{l=1}^{n} w_i(t-\tau_{kl}) \\
&+ \sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \\
&+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t).
\end{aligned}
$$

Moreover, $\dot{V}_2 = \sum_{l=1}^{n}\int_{-\tau_{kl}}^{0}[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)} + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) + (\Phi_i^T \otimes I_n)F(t)]^T Z_i[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)} + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) + (\Phi_i^T \otimes I_n)F(t)]d\alpha - \sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha$.
$\dot{V}_2(w_i(t),t) \leq nh[D\bar{f}(s(t))\omega_i(t)+c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]^T Z_i[D\bar{f}(s(t))\omega_i(t)+c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]$ $+2nh(D\bar{f}(s)\omega_i)^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega + 2nh(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)F(t)$ $+2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega + 2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)F(t) + 2nh((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega)^T Z_i(\Phi_i^T \otimes I_n)F + nh((\Phi_i^T \otimes I_n)F(t))^T Z_i((\Phi_i^T \otimes I_n)F(t)) - \sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha + nh((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega)^T Z_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega)$.
$\dot{V}_3(w_i(t),t) \leq nw_i^T(t)Q_i w_i(t) - \sum_{l=1}^{n} w_i^T(t-\tau_k)Q_i w_i(t-\tau_{kl})$.
Then, we have $\sum_{k=1}^{3}\dot{V}_k(w_i(t),t) \leq w_i^T[\dot{P}_i(t) + P_i(t)D\bar{f}(s) + D\bar{f}(s)^T P_i(t) + nhX_i +$

$2n(Y_i - \frac{c\lambda_i}{n}P_i(t)\Gamma)]w_i(t) + 2w_i^T(t)(\frac{c\lambda_i}{n}P_i(t)\Gamma - Y_i)\sum_{l=1}^{n} w_i(t-\tau_{kl}) + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) + 2nh(D\bar{f}(s)\omega_i)^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega + nh[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]^T Z_i[D\bar{f}(s(t))\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}] + 2nh(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)F(t) + 2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega + 2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)F + 2nh((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i(\Phi_i^T \otimes I_n)F(t) + nh((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)) + nh((\Phi_i^T \otimes I_n)F(t))^T Z_i((\Phi_i^T \otimes I_n)F(t)) + nw_i^T Q_i w_i - \sum_{l=1}^{n} w_i^T(t-\tau_{kl})Q_i w_i(t-\tau_{kl})$.

Applying the Young Inequality gives $2nh(c\lambda_i\Gamma\overline{w_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \leq n^2h^2c^2\lambda_i^2 w^T((\Phi \otimes I_n)^T I(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Pi_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n))w(t) + \overline{w_i(t-\tau_j)}^T \Pi_i^{-1}\overline{w_i(t-\tau_j)}$

and

$2nh(c\lambda_i\Gamma\overline{w_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)F(t) \leq n^2h^2c^2\lambda_i^2 F^T(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Theta_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)F(t) + \overline{w_i(t-\tau_j)}^T \Sigma_i^{-1}\overline{w_i(t-\tau_j)}$.

Applying these results to the inequality, then we have

$$\begin{aligned}
\dot{V} &\leq \sum_{i=2}^{N} \begin{pmatrix} w_i(t) \\ \overline{w_i(t-\tau_j)} \end{pmatrix}^T \Xi \begin{pmatrix} w_i(t) \\ \overline{w_i(t-\tau_j)} \end{pmatrix} + 2\mu(t)\beta \\
&+ (\|w\|(2\gamma\beta + 2nh\gamma\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + 2nh\mu(t)\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i) \\
&+ nh\gamma^2\sum_{i=2}^{N}\lambda_{max}(Z_i) + n^2h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) \\
&+ (nhc\mu)^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}\Theta_i\lambda_i^2\lambda_{max}^2 Z_i)\|w\| + 2nh\gamma\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)) \\
&+ nh\mu^2(t)\sum_{i=2}^{N}\lambda_i^2\lambda_{max}(Z_i). \qquad (7.32)
\end{aligned}$$

Thus when

$$\|w\| \geq \frac{2\mu(t)\beta + 2nh\gamma\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)}{\varpi(t)},$$

we have

$$\dot{V} \leq -\delta\|w\|^2 + nh\mu^2(t)\sum_{i=2}^{N}\lambda_{max}(Z_i)\lambda_i^2, \tag{7.33}$$

where Ξ is defined in (**??**),

$$\begin{aligned}\varpi(t) = &-(2\gamma\beta + 2nh\gamma\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + 2nh\mu\|\Sigma_i\|\sum_{i=2}^{N}\lambda_{max}(Z_i)\\ &+ n^2h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) + nh\gamma^2\sum_{i=2}^{N}\lambda_{max}(Z_i)\\ &+ h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Theta_i)\lambda_i^2\lambda_{max}^2(Z_i)) - \delta.\end{aligned} \tag{7.34}$$

Then, according to definition 4.1 and Lyapunov stability theory, bounded consensus is ultimately achieved.

Remark 7.4. The above two bounded consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical nodes to the cases of non-identical nodes. Because of the complexity of the consensus problems for non-identical nodes, we only obtain here sufficient conditions instead of sufficient and necessary conditions. At the same time, the conditions obtained here are somewhat complicated and difficult to verify, but according to certain specific cases, we can construct an appropriate numerical simulation example to verify the effectiveness of the proposed results. Comparing the above two theorems, it can be seen that the boundary of the convergence set and the maximum size of time delay can be evaluated respectively.

Now we'll investigate the global bounded consensus problem for the following NMAS which can be viewed as a special case of NMAS (7.1):

$$\dot{x}_i(t) = f_i(x_i(t)) + u_i, \quad i = 1, 2, \cdots, N, \tag{7.35}$$

where all parameters have the same meanings as those in (7.1), and the unique difference is that every node has the same retardation time vector $(\tau_1, \tau_2, \cdots, \tau_n)$. Repeating the similar process, we can transfer the consensus problem of NMAS to the stability problem of the $N-1$ of $n-$dimensional systems:

$$\begin{aligned} \dot{\omega}_i(t) = {} & D\bar{f}(s(t)\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau)} + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\ & + (\Phi_i^T \otimes I_n)F(t), \ i = 2, 3, \cdots, N. \end{aligned} \tag{7.36}$$

Similar to the analysis of the Theorem 7.1, one can get the following corollary easily:

Corollary 7.1 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$ and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$\begin{aligned} & a\|x(t)\|^2 \leq x^T(t)P_i(t)x(t) \leq b\|x(t)\|^2, \\ & \qquad \forall t \in R^+, \ x \in R^n, i = 2, 3, \cdots, N, \end{aligned} \tag{7.37}$$

$$\mu_2(\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma) + \zeta I \leq 0, \quad i = 1, 2, \cdots, N, \tag{7.38}$$

$$\|I(t)\| \leq \gamma, \ i = 1, 2, \cdots, N. \tag{7.39}$$

and exist an i and a j such that

$$\begin{pmatrix} -\lambda_j(\Gamma+\Gamma^T) & \lambda_i\Gamma \\ \lambda_i\Gamma^T & 0 \end{pmatrix} \le 0, \tag{7.40}$$

for all $t \ge t_0$, where $i = 2, 3, \cdots, n$, $j = 1, 2, \cdots, n$.
Let

$$\mu(t) = \|F(t)\| \tag{7.41}$$

be bounded and

$$\beta = (\sum_{i=2}^{N} \|P_i(t)\|^2)^{\frac{1}{2}},$$

if $\zeta > 2\gamma\beta$, then the system (7.46) converges to the set

$$M = \{e(t) | \|e(t)\| \le \frac{2b}{a} \frac{\beta \overline{lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\}, \tag{7.42}$$

namely, $e(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^{N} x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, and then the NMAS (7.1) achieves bounded consensus for any fixed time delay $\tau_k > 0 (k = 1, 2, \cdots, n)$.

7.4 Global Controlled Bounded Consensus Criterion

7.4.1 Linear Feedback Pinning Controller

To achieve the goal, we apply the feedback control strategy on a small fraction δ $(0 < \delta \le 1)$ of the agents in system (7.1). Suppose that nodes $i_1, i_2, \cdots, i_l$ are selected to be under control, where $l = [\delta N]$ represents the smaller but nearest integer to the real number δN. This controlled MAS can be described as

$$\begin{cases} \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j\in\mathcal{N}_i}^{N} a_{i_k j}\Gamma\overline{x_j(t-\tau_j)} + u_{i_k}, & 1 \le k \le l, \\ \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j\in\mathcal{N}_i}^{N} a_{i_k j}\Gamma\overline{x_j(t-\tau_j)}, & l+1 \le k \le N. \end{cases} \tag{7.43}$$

The local linear negative feedback control law is chosen as follows:

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k} - s), & 1 \le k \le l, \\ u_{i_k} = 0, & l+1 \le k \le N, \end{cases} \tag{7.44}$$

where the feedback gain $d_{i_k} > 0$.

Combine (7.43) and (7.44) and rearrange the order of the nodes in the network. Let the first l nodes be controlled, and $e_i = x_i - s,\ i = 1, 2, \cdots, N$. It's obvious that $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i}^{N} a_{kj}\Gamma\overline{x_j(t-\tau_j)} = 0$ and $\sum_{i=1}^{N} e_i = 0$. Then by applying the Newton-Leibniz formula, error systems can be written as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma\overline{e_j(t-\tau_j)} + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s) - d_i e_i, 1 \le i \le l, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma\overline{e_j(t-\tau_j)} + \int_0^1 (Df_i + \tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s), l+1 \le i \le N. \end{cases} \tag{7.45}$$

The following work will focus on simplifying the error systems (7.45) by means of a series of transformations using a procedure similar to [88].

Define the following matrix

$$D = diag(D_1, D_2, \cdots, D_N) \in R^{nN\times nN},$$

where $D_i = diag\{-d_i, -d_i, \cdots, -d_i\} \in R^{n\times n}$.

Let $e = (e_1^T, e_2^T, \cdots, e_N^T)^T$, then (7.45) becomes

$$\dot{e} = \bar{\Sigma}(t)e + cA \otimes \Gamma e(t-\tau) + I(t)e - \frac{1}{N}H(t)e + F(t), \tag{7.46}$$

where $\bar{\Sigma}(t) = I_N \otimes D\bar{f}(s) + D$, $H^T(t) = (H_1^T(t), \cdots, H_N^T(t))$, $H_i(t) = (\int_0^1 Df_1(s+\tau e_1)d\tau, \cdots, \int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t) = (f_1^T(s) - \bar{f}^T(s), \cdots, f_N^T(s) - \bar{f}^T(s))$ and $I(t) = diag\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau, \cdots, \int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\}$.

Since A is symmetric and irreducible, according to [88], there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \cdots, \Phi_N)$. This together with $w(t) = (\Phi^T \otimes I_n)e$ gives

$$\begin{aligned}\dot{w} &= (\Phi^T \otimes I_n)\bar{\Sigma}(t)(\Phi \otimes I_n)w + (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)\overline{w(t-\tau_j)} \\ &\quad + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w - \frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)w + (\Phi^T \otimes I_n)F(t).\end{aligned} \tag{7.47}$$

Note that $H(t) = \sqrt{N}\sum_{k=1}^N (\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ represents the matrix with its k-th column equal to Φ_1 and the remaining elements are zero. Then we have $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^N (\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau(\Phi \otimes I_n)$, where I_k represents the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the remaining of its elements are zero.

Then $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^N \begin{pmatrix} \Upsilon_k \\ \mathsf{0} \end{pmatrix} \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k \in R^{1\times N}$ and $\mathsf{0} \in R^{(N-1)\times N}$. Therefore, $\dot{w} = \bar{\Sigma}(t)w + c\Lambda \otimes \Gamma\overline{w(t-\tau_j)} + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w - \begin{pmatrix} * \\ \mathsf{0} \end{pmatrix} w + (\Phi^T \otimes I_n)F(t)$. Since $w_1 \equiv 0$, we only need to consider $w_2, w_3, \cdots, w_N$. Rewriting in the component form we have

$$\dot{w}_i = \tilde{\Sigma}_i(t)w_i + c\lambda_i\Gamma\overline{w_i(t-\tau_j)} + (\Phi_i^T \otimes I_n)F(t) + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w,$$
$$i = 2, 3, \cdots, N, \tag{7.48}$$

where $\tilde{\Sigma}_i = \bar{D}f(s) + D_i$.

Theorem 7.3 Suppose there exist matrices $P_i(t) > 0$, $Q_i > 0$, $\Pi_i > 0$, $\Sigma_i > 0$, X_i, Y_i and Z_i of appropriate dimensions and constant $\gamma \geq 0$ such that $\|I(t)\| \leq \gamma$ and

$$\Xi = \begin{pmatrix} \Xi_{11} & \dfrac{c\lambda_i}{n}P_i(t)\Gamma - Y_i + c\lambda_i nh\tilde{\Sigma}_i(t)^T Z_i\Gamma \\ * & \Pi_i^{-1} + \Sigma_i^{-1} - Q_i + c^2\lambda_i^2 nh\Gamma^T Z_i\Gamma \end{pmatrix} < 0, \tag{7.49}$$

where $\Xi_{11} = \dot{P}_i(t) + P_i(t)\tilde{\Sigma}_i(t) + \tilde{\Sigma}_i(t)^T P_i(t) + nhX_i + nQ_i + 2n(Y_i - \frac{c\lambda_i}{n}P_i(t)\Gamma) + nh\tilde{\Sigma}_i(t)^T Z_i\tilde{\Sigma}_i(t)$

$$\begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{7.50}$$

for $i = 2, 3, \cdots, N$, $\mu(t) = \|F(t)\|$ is bounded and $\beta = (\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}}$. Then the NMAS (7.1) achieves global bounded consensus for any fixed time delays $\tau_{kl} \in [0, h] > 0(k, l = 1, 2, \cdots, n)$ for some $h < \infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V(w_i(t), t) = \sum_{i=2}^{N}\sum_{k=1}^{3} V_k(w_i(t), t), \tag{7.51}$$

where

$$
\begin{aligned}
V_1(w_i(t),t) &= w_i^T(t)P_i(t)w_i(t),\\
V_2(w_i(t),t) &= \sum_{l=1}^{n}\int_{-\tau_{kl}}^{0}\int_{t+\beta}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha d\beta,\\
V_3(w_i(t),t) &= \sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}w_i^T(\alpha)Q_iw_i(\alpha)d\alpha.
\end{aligned}
$$

The i-th ($i = 2, 3, \cdots, N$) equation in system (7.48) can be written as

$$
\begin{aligned}
\dot{\omega}_i(t) &= (\tilde{\Sigma}_i(t) + c\lambda_i\Gamma)\omega_i(t) - \frac{c\lambda_i}{n}\Gamma\sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i(\alpha)d\alpha\\
&\quad + (\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t) + (\Phi_i^T\otimes I_n)F(t), \quad i = 2, 3, \cdots, N, \quad (7.52)
\end{aligned}
$$

and, thus, the derivative of $V_1(w_i(t),t)$ satisfies

$$
\begin{aligned}
\dot{V}_1(w_i(t),t) &= w_i^T(t)(\dot{P}_i(t) + P_i(t)(\tilde{\Sigma}_i(t) + c\lambda_i\Gamma) + (\tilde{\Sigma}_i(t) + c\lambda_i\Gamma)^TP_i(t))w_i(t)\\
&\quad - \frac{2c\lambda_i}{n}w_i^T(t)P_i(t)\Gamma\sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i(\alpha)d\alpha\\
&\quad + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)F(t).
\end{aligned}
\quad (7.53)
$$

Defining $a(.)$, $b(.)$ and M as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$ and $M = \frac{c\lambda_i}{n}P_i(t)\Gamma$ for all $\alpha \in [t-\tau_{kl}, t]$ and then applying Lemma 4.2 results in $\dot{V}_1(w_i(t),t) \le w_i^T(t)[\dot{P}_i(t) + P_i(t)\tilde{\Sigma}_i(t) + \tilde{\Sigma}_i(t)^TP_i(t) + nhX_i + 2n(Y_i - \frac{c\lambda_i}{n}P_i(t)\Gamma)]w_i(t) + 2w_i^T(t)(\frac{c\lambda_i}{n}P_i(t)\Gamma - Y_i)\sum_{l=1}^{n}w_i(t-\tau_{kl}) + \sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w(t) + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)F(t)$.

Moreover, $\dot{V}_2(w_i,t) \le nh[\tilde{\Sigma}_i\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]^TZ_i[\tilde{\Sigma}_i(t)\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}] + 2nh(\tilde{\Sigma}_i(t)\omega_i(t))^TZ_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t) + 2nh(\tilde{\Sigma}_i(t)\omega_i(t))^TZ_i(\Phi_i^T\otimes I_n)F(t) +$

$2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) + 2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)F + 2nh((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i(\Phi_i^T \otimes I_n)F + nh((\Phi_i^T \otimes I_n)F)^T Z_i((\Phi_i^T \otimes I_n)F(t)) - \sum_{l=1}^n \int_{t-\tau_{kl}}^t \dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha + nh((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))$ and $\dot{V}_3(w_i(t),t) \leq nw_i^T(t)Q_iw_i(t) - \sum_{l=1}^n w_i^T(t-\tau_k)Q_iw_i(t-\tau_{kl})$.

The derivative of $V(w_i(t),t)$ is

$$
\begin{aligned}
&\sum_{k=1}^{3} \dot{V}_k(w_i(t),t)\\
&\leq w_i^T(t)[\dot{P}_i(t) + P_i(t)\tilde{\Sigma}_i(t) + \tilde{\Sigma}_i(t)^T P_i(t) + nhX_i + 2n(Y_i - \frac{c\lambda_i}{n}P_i(t)\Gamma)]w_i(t)\\
&+ 2w_i^T(\frac{c\lambda_i}{n}P_i(t)\Gamma - Y_i)\sum_{l=1}^{n} w_i(t-\tau_{kl}) + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w\\
&+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) + 2nh(\tilde{\Sigma}_i(t)\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)\\
&+ nh[\tilde{\Sigma}_i(t)\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]^T Z_i[\tilde{\Sigma}_i(t)\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]\\
&+ 2nh(\tilde{\Sigma}_i(t)\omega_i(t))^T Z_i(\Phi_i^T \otimes I_n)F(t)\\
&+ 2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)\\
&+ 2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)F(t)\\
&+ 2nh((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i(\Phi_i^T \otimes I_n)F(t)\\
&+ nh((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))^T Z_i((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t))\\
&+ nh((\Phi_i^T \otimes I_n)F(t))^T Z_i((\Phi_i^T \otimes I_n)F(t)) + nw_i^T(t)Q_iw_i(t)\\
&- \sum_{l=1}^{n} w_i^T(t-\tau_{kl})Q_iw_i(t-\tau_{kl}).
\end{aligned}
$$

Applying the Young Inequality, then we have

$$2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \leq \overline{w_i(t-\tau_j)}^T \Pi_i^{-1}\overline{w_i(t-\tau_j)}$$
$$+ n^2h^2c^2\lambda_i^2 w^T(t)((\Phi \otimes I_n)^T I(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Pi_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n))w, \tag{7.54}$$

and

$$2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T \otimes I_n)F(t)$$
$$\leq \overline{w_i(t-\tau_j)}^T \Sigma_i^{-1}\overline{w_i(t-\tau_j)} + n^2h^2c^2\lambda_i^2 F^T(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Sigma_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)F(t). \tag{7.55}$$

Applying theorem's conditions results:

$$\dot{V}(w_i(t),t) \leq \sum_{i=2}^{N}\left(\begin{array}{c} w_i(t) \\ \hline w_i(t-\tau_j)\end{array}\right)^T \Xi \left(\begin{array}{c} w_i(t) \\ \hline w_i(t-\tau_j)\end{array}\right)$$
$$+ \|w\|((2\gamma\beta + 2nhr\|\tilde{\Sigma}_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + 2nh\mu(t)\|\tilde{\Sigma}_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i)$$
$$+ nhr^2\sum_{i=2}^{N}\lambda_{max}(Z_i) + n^2h^2c^2r^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i)$$
$$+ h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Sigma_i)\lambda_i^2\lambda_{max}^2(Z_i))\|w\|$$
$$+ 2\mu(t)\beta + 2nhr\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)) + nh\mu^2(t)\sum_{i=2}^{N}\lambda_{max}(Z_i). \tag{7.56}$$

Thus when

$$\|w\| \geq \frac{2\mu(t)\beta + 2nhr\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)}{\begin{array}{c}-(2\gamma\beta + 2nhr\|\tilde{\Sigma}_i(t)\|\sum\limits_{i=2}^{N}\lambda_{max}(Z_i) + 2nh\mu(t)\|\tilde{\Sigma}_i(t)\|\sum\limits_{i=2}^{N}\lambda_{max}(Z_i) \\ +nhr^2\sum\limits_{i=2}^{N}\lambda_{max}(Z_i) + n^2h^2c^2r^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum\limits_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) \\ +h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum\limits_{i=2}^{N}\lambda_{max}(\Sigma_i)\lambda_i^2\lambda_{max}^2(Z_i)) - \delta\end{array}},$$

we have

$$\dot{V} \leq -\delta\|w\|^2 + nh\mu^2(t)\sum_{i=2}^{N}\lambda_{max}(Z_i), \tag{7.57}$$

where Ξ is defined in (7.61). Then, according to definition 4.1 and Lyapunov stability theory, bounded consensus is ultimately achieved.

7.4.2 Adaptive Pinning Controller

In this section, we will derive globally consensus criteria via direct adaptive pinning control method. Without loss of generality, we still assume that the first l agents are selected as pinned agents with the adaptive controllers:

$$\begin{cases} u_i = -d_i(t)(x_i - s), & 1 \leq i \leq l, \\ \dot{d}_i(t) = h_i e_i^T P_i(t) e_i, & \\ u_i = 0, & l+1 \leq i \leq N, \end{cases} \tag{7.58}$$

where constant $h_i > 0$ and positive definite matrix $P_i(t) \in R^{n\times n}$. Applying Newton-Leibniz formula, then the error NMAS can be rewritten as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma\overline{e_j(t-\tau_j)} + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \qquad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s) - d_i(t)e_i,\ 1\le i\le l, \\ \dot{d}_i(t) = h_i e_i^T P_i(t) e_i, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma\overline{e_j(t-\tau_j)} + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \qquad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s), \quad l+1\le i\le N. \end{cases} \tag{7.59}$$

Repeating a similar procedure to the previous subsection, the controlled consensus problem of system (7.1) is equivalent to the stability problem of the following $N-1$ of n-dimensional systems.

$$\begin{cases} \dot{w}_i = D\bar{f}(s(t))w_i - d_i(t)w_i + c\lambda_i\Gamma\overline{w_i(t-\tau_j)} + (\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w \\ \qquad +(\Phi_i^T\otimes I_n)F(t), \qquad\qquad 2\le i\le l, \\ \dot{d}_i(t) = h_i w_i^T P_i(t) w_i, \\ \dot{w}_i = D\bar{f}(s)w_i + c\lambda_i\Gamma\overline{w_i(t-\tau_j)} + (\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w \\ \qquad +(\Phi_i^T\otimes I_n)F(t), \qquad\qquad l+1\le i\le N, \end{cases} \tag{7.60}$$

where w_i, w, Φ, Φ_i, $I(t)$ and $F(t)$ are the same as the previous subsection.

Theorem 7.4 Suppose there exist matrices $P_i(t) > 0$, $Q_i > 0$, $\Pi_i > 0$, $\Sigma_i > 0$, X_i, Y_i and Z_i of appropriate dimensions and constant $\gamma \ge 0$ such that $\|I(t)\| \le \gamma$

and

$$\Xi = \begin{pmatrix} \Xi_{11} & \dfrac{c\lambda_i}{n}P_i(t)\Gamma - Y_i + c\lambda_i nh\tilde{\Sigma}_i(t)^T Z_i\Gamma \\ * & \Pi_i^{-1} + \Sigma_i^{-1} - Q_i + c^2\lambda_i^2 nh\Gamma^T Z_i\Gamma \end{pmatrix} < 0, \tag{7.61}$$

where $\Xi_{11} = \dot{P}_i(t) + P_i(t)(Df(s(t))) + (Df(s(t)))^T P_i(t) - 2dP_i(t) + nhX_i + nQ_i + 2n(Y_i - \frac{c\lambda_i}{n}P_i(t)\Gamma) + nh\tilde{\Sigma}_i(t)^T Z_i\tilde{\Sigma}_i(t)$

$$\begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{7.62}$$

for $i = 2, 3, \cdots, N$, $\mu(t) = \|F(t)\|$ is bounded and $\beta = (\sum_{i=2}^N \|P_i(t)\|^2)^{\frac{1}{2}}$. Then the NMAS (7.1) achieves global bounded consensus for any fixed time delays $\tau_{kl} \in [0, h] > 0 (k, l = 1, 2, \cdots, n)$ for some $h < \infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V(w_i(t), t) = \sum_{i=2}^N \sum_{k=1}^3 V_k(w_i(t), t) + \sum_{i=2}^l \frac{(d_i(t) - d)^2}{h_i}, \tag{7.63}$$

where

$$\begin{aligned} V_1(w_i(t), t) &= w_i^T(t)P_i(t)w_i(t), \\ V_2(w_i(t), t) &= \sum_{l=1}^n \int_{-\tau_{kl}}^0 \int_{t+\beta}^t \dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha d\beta, \\ V_3(w_i(t), t) &= \sum_{l=1}^n \int_{t-\tau_{kl}}^t w_i^T(\alpha)Q_i w_i(\alpha)d\alpha. \end{aligned}$$

The remaining part of the proof is similar to that of Theorem 7.1 and is therefore omitted here. This completes the proof.

7.5 An Example

In this section, a NMAS consisting of 11 non-identical agents will be constructed to demonstrate efficiency of the results proposed in the previous section. For simplification, the delay-dependent result will be verified solely and the delay-independent result can be verified similarly. The objective is to guarantee 11 agents achieve bounded consensus, and the consensus curves are described as the average dynamics of 11 agents in a 2-dimensional coordinate system.

The agent dynamics can be chosen as follows

$$\dot{x}_i(t) = B_i x_i(t) + g(x_i(t)), \quad i = 1, 2, \cdots, 11, \tag{7.64}$$

where

$$\begin{cases} B_i = \begin{pmatrix} -10 + 0.1 \times (i-1) & 10 - 0.1 \times (i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15 - 0.1 \times (i-1) & 0 \end{pmatrix}, & i = 1, 2, \cdots, 6, \\ B_i = \begin{pmatrix} -10 - 0.1 \times (i-6) & 10 + 0.1 \times (i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15 + 0.1 \times (i-6) & 0 \end{pmatrix}, & i = 7, 8, \cdots, 11, \end{cases}$$

and

$$g(x_i(t)) = (-9.5 sin(\frac{\pi x_{i1}(t)}{3.2} + \pi) \ 0 \ 0)^T, \quad i = 1, 2, \cdots, 11.$$

The communication coupling matrix A and the inner coupling matrix are

$$A = \begin{pmatrix} -8 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -8 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -6 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & -7 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -5 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -10 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -6 \end{pmatrix}, \Gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

respectively.

Given the initial values of 11 agents as $(10\ 5\ -10)^T$, $(12\ 6\ -12)^T$, $(14\ 7\ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$ and $(-10\ 15\ 10)^T$ respectively, for any time delays $0 < \tau_{kl} \leq 0.061$, $1 \leq k, l \leq 11$, the NMAS will achieve consensus by means of verifying the conditions of Theorem 7.2 readily. The simulation results of the consensus errors are depicted in Fig 7.1, Fig 7.2 and Fig 7.3 for $c = 1$, and the communication time delay matrix used to produce the present simulation curves are given as follows:

$$
\begin{array}{c}
\begin{pmatrix}
0.01 & 0.02 & 0.03 \\
0.04 & 0.05 & 0.06 \\
0.015 & 0.016 & 0.017 \\
0.018 & 0.019 & 0.0195 \\
0.021 & 0.022 & 0.023 \\
0.024 & 0.025 & 0.026 \\
0.027 & 0.028 & 0.029 \\
0.031 & 0.032 & 0.033 \\
0.034 & 0.035 & 0.036 \\
0.037 & 0.038 & 0.039 \\
0.045 & 0.055 & 0.061
\end{pmatrix}
\end{array}
\begin{array}{l}
\leftarrow\ 1st\ agent \\
\leftarrow\ 2nd\ agent \\
\leftarrow\ 3rd\ agent \\
\leftarrow\ 4th\ agent \\
\leftarrow\ 5th\ agent \\
\leftarrow\ 6th\ agent \\
\leftarrow\ 7th\ agent \\
\leftarrow\ 8th\ agent \\
\leftarrow\ 9th\ agent \\
\leftarrow\ 10th\ agent \\
\leftarrow\ 11th\ agent
\end{array}
$$

7.6 Conclusions

This chapter investigates the global consensus problems of NMAS with different agent dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on a series of transformations and Lyapunov stability theorem. The communication connection between agents are not direct, and there are different constant time delays in the communication topology. It should be noted that the conditions are still restrictive. Further investigations will focus on relaxing these limitations.

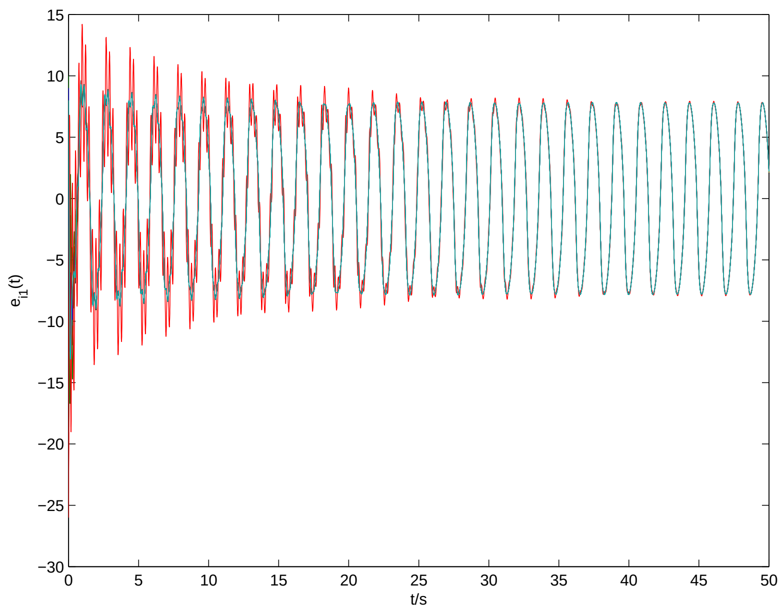

Fig 7.1 The consensus errors $e_{i1}(t)$.

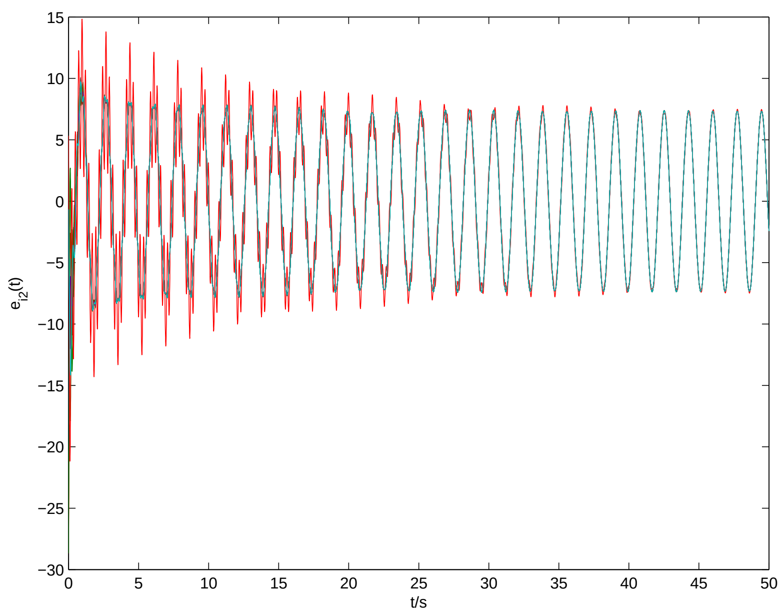

Fig 7.2 The consensus errors $e_{i2}(t)$.

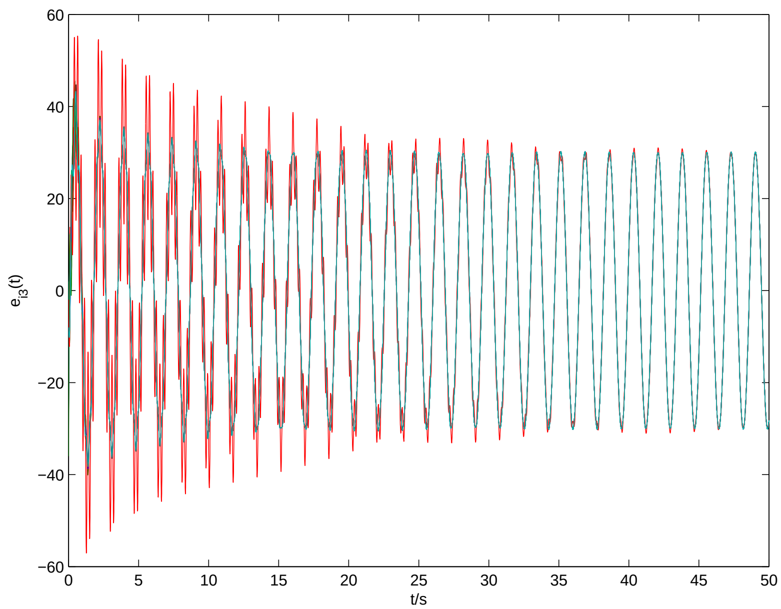

Fig 7.3 The consensus errors $e_{i3}(t)$.

Chapter 8

Conclusions

8.1 Conclusions

In the thesis, controlled consensus of NMAS with identical and non-identical agent dynamics have been investigated, several controlled consensus criteria have been obtained and all related results have been demonstrated effective by means of numerical simulations. The main contributions of the present thesis are summarized as follows.

1. Exact consensus of NMAS with nonlinear agent dynamics and communication delay

The average dynamics of all agents have been used as the desired moving trajectories, and then linear feedback and adaptive feedback have been presented to guarantee its global exact consensus based on the Lyapunov stability theory. The controllers designed here are relatively simple in form, but are effective to resolve the consensus problem of the NMAS with nonlinear agent dynamics and communication delay. It should be noted that the conditions obtained here are still restrictive.

2. Exact consensus of NMAS with uncertain coupling structure

The local and global consensus problems of NMAS with uncertain coupling structure has been investigated. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on a series of transformations and Lyapunov stability theorem. Compared with many existing results, the controllers obtained here are more effective in resolving the consensus problem with uncertain coupling structure.

3. Global bounded consensus of NMAS with non-identical agent dynamics

The global consensus problems of NMAS with non-identical agent dynamics have been discussed, and the proposed consensus criterion is formulated in terms of certain boundedness of state errors. In addition, the exact consensus are also investigated by means of nonlinear controllers. Compared with many existing results, the results obtained here make two significant advances. One is the related results for the case of identical agent dynamics has been generalized to the case of non-identical agent dynamics; the other is that the nonlinear feedback controllers designed here can guarantee the NMAS achieve exact consensus.

4. Global bounded consensus of NMAS with nonlinear, non-identical agent dynamics and communication time-delay

The thesis investigates the global bounded consensus problem of NMAS exhibiting nonlinear, non-identical agent dynamics and communication time-delay. Globally bounded consensus conditions for both delay-independent and delay-dependent conditions based on the Lypunov-Krasovskii functional method are derived. In addition, globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived. The proposed consensus criteria ensures that all agents eventually move along the desired trajectories in the sense of boundedness. The proposed consensus criteria generalizes the case of identical agent dynamics to the case of non-identical agent dynamics and many related results in this area can be viewed as special cases of the present results. Finally, the effectiveness of the theoretical results will be demonstrated by means of numerical simulations.

5. Global bounded consensus of NMAS with different agent dynamics and time-varying delay topology

The thesis also investigats the consensus problems of NMAS with nonlinear, non-identical agent dynamics and communication time-varying delay. The consensus for the NMAS is achieved based on a series of transformations and Lyapunov stability theorem, the proposed results have also been extended to the controlled consensus problems based on pining control and adaptive pining control scheme. Many related results for the case of identical agent dynamics have been viewed as the special cases of the proposed results. However, it should be noted that the conditions are still restrictive and the time-varying delay is chosen as special case.

6. Global controlled bounded consensus of NMAS with nonlinear, non-identical agent dynamics and multiple delay topology

The thesis finally investigates the global bounded consensus problem of NMAS consisting of nonlinear, non-identical node dynamics with multiple delays. The globally bounded controlled consensus conditions for both delay-independent and delay-dependent conditions based on the Lypunov-Krasovskii functional method have been derived. The proposed consensus criteria ensures that all agents eventually move along the desired trajectory in the sense of boundedness. The effectiveness of the theoretical results have been demonstrated by means of numerical simulations.

8.2 Future Work

The problem of consensus on NMAS of structurally different dynamical agent is investigated in the present thesis. Consensus of NMAS is usually defined in terms of identical accordance on the evolution of each individual agent in the network. However, for a network consisting of strictly different agents, this type of consensus should be redefined. In this case, a generalized definition of consensus can be considered, where the evolution of each agent can be related to others in terms of a map. In order to achieve consensus on a network of strictly different agents, local linear and nonlinear controllers, adaptive and pinning adaptive controllers are designed which force the network to achieve bounded consensus or exact consensus respectively. However, the conditions obtained in this paper are still restrictive for verification, thus further research will focus on these problems.

1. Output consensus

Output consensus problems will be considered in the further research, compared with the state consensus problem, output consensus problem is a bit more complicated because the the corresponding proof has to turn to the Lasalle's Invariance Principle and the Cauchy's convergence criteria. Some relatively simple results have been gotten, but need further systematic investigation on this problem.

2. Nonlinear constructive methods

Feasibility study to NMAS based on the nonlinear constructive methods, such as backstepping, forwarding and interlacing. If possible, the corresponding Lyapunov functional constructing process may be simplified increasingly. Some NMAS with special topology can be transformed into the necessary forms when using such methods, thus further research will focus on how to expand their application scales.

3. Switching topology

Most of the existing works have discussed static agent networks whose coupling matrices are constant in time. However, the interaction of connected nodes in some real-world agent networks may change abruptly. This kind of agent networks can be modeled as the NMAS with switching topology. But, for switched agent network, not enough attention has been deserved, and very few works can be found in the literature. So many open problems of such NMAS are remained to be solved.

4. Nonlinear topology structure

For the sake of simplification, many NMAS coupling terms have been assumed to be linear instead of nonlinear. Such kind of topology structure is difficult to describe by means of the graph theory, thus to search for more useful modeling and disposal methods are very meaningful. Some simple nonlinear coupling results have been reported, but the results are too conservatism to applicable universally.

5. Other issues

Issues like disturbances, time delay, communication noise, sensor noise, and model uncertainties need to be taken into account. Future research may be involved in studying how communication noise and inconsistent time-delay from different neighboring agents affect consensus for the whole system under dynamically changing information exchange topologies.

Bibliography

[1] A. Chapman, M. Mesbahi. System Theoretic Aspects of Influenced Consensus: Single Input Case[J].*IEEE Transactions on Automatic Control*, 2012, 57(6): 1505 – 1511. 1

[2] M.H. Zhu and S. Martinez. On Distributed Convex Optimization Under Inequality and Equality Constraints[J].*IEEE Transactions on Automatic Control*, 2012, 57(1): 151 – 164. 1

[3] D.V. Dimarogonas, E. Frazzoli and K.H. Johansson. Distributed Event-Triggered Control for Multi-Agent Systems[J].*IEEE Transactions on Automatic Control*, 2012, 57(5): 1291 – 1297. 1

[4] J.Z. Li, W. Ren and S.Y. Xu. Distributed Containment Control with Multiple Dynamic Leaders for Double-Integrator Dynamics Using Only Position Measurements[J].*IEEE Transactions on Automatic Control*, 2012, 57(6): 1553 – 1559. 1

[5] U. Munz, A.Papachristodoulou and F. Allgower. Delay Robustness in Non-Identical Multi-Agent Systems[J].*IEEE Transactions on Automatic Control*, 2012, 57(6): 1597 – 1603. 1

[6] L. Busoniu, R. Babuska and B.D. Schutter. A comprehensive survey of multi-

agent reinforcement learning[J]. *IEEE Transactions on Systems, Man, and Cybernetics. Part C: Applications and Reviews*, 2008, 38(2):156 – 172. 1

[7] H.J. Fang, Z.H. Wu and J. Wei. Improvement for Consensus Performance of Multi-Agent Systems Based on Weighted Average Prediction[J]. *IEEE Transactions on Automatic Control*, 2012, 57(1):249 – 252. 1

[8] G.D. Shi, Y.G. Hong and K.H. Johansson. Connectivity and Set Tracking of Multi-Agent Systems Guided by Multiple Moving Leaders[J]. *IEEE Transactions on Automatic Control*, 2012, 57(3):663 – 676. 1

[9] Hongkeun Kim, Hyungbo Shim and Jin Heon Seo. Output Consensus of Heterogeneous Uncertain Linear Multi-Agent Systems[J]. *IEEE Transactions on Automatic Control*, 2011, 56(1):200 – 206. 1

[10] A. Jadbabaie, J. Lin and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules[J]. *IEEE Transactions on Automatic Control*, 2003, 48(6): 988 -C 1001. 1

[11] L. Wang and G. Xie. Consensus control for a class of networks of dynamic agents[J]. *Int. J. Robust Nonlin. Control*, 2007, 17(10C11):941 C- 959. 1

[12] H. G. Tanner and D. K. Christodoulakis. State synchronization in local-interaction networks is robust with respect to time delays[C]. *in Proc. IEEE Conf. Decision Control, Eur. Control Conf.*, 2005, 4945 -C 4950. 1

[13] F. Fagnani and S. Zampieri. Average consensus with packet drop communication[C]. *in Proc. 45th IEEE Conf. Decision Control*, San Diego, CA, Dec. 2006, pp. 1007C1012. 1

[14] D. B. Kingston and R. W. Beard. Discrete-time average-consensus under

switching network topologies[C]. *in Proc. Amer. Control Conf.*, Minneapolis, MN, Jun. 2006, pp. 3551C3556. 1

[15] A. Sinha and D. Ghose. Generalization of linear cyclic pursuit with application to rendezvous of multiple autonomous agents[J]. *IEEE Transactions on Automatic Control*, 2006, 51(11): 1819 -C 1824. 1

[16] T. Li and J. F. Zhang. Decentralized tracking-Type games for multiagent systems with coupled ARX models: Asymptotic nash equilibria[J]. *Automatica*, 2008, 44(3): 713 -C 725. 1

[17] A. Kashyap, T. Basar and R. Srikant. Quantized consensus[J]. *Automatica*, 2007, 43(7): 1192 -C 1203. 1

[18] R. Carli, F. Fagnani, A. Speranzon and S. Zampieri. Communication constraints in coordinated consensus problems[C]. *in Proc. Amer. Control Conf.*, Minneapolis, MN, Jun. 2006, pp. 4189 C- 4194. 1

[19] A. Olshevsky and J. N. Tsitsiklis. Convergence rates in distributed consensus and averaging[C]. *in Proc. 45th IEEE Conf. Decision Control*, San Diego, CA, 2006, pp. 3387C3392. 1

[20] V. D. Blondel, J. M. Hendrickx, A. Olshevsky and J. N. Tsitsiklis. Convergence in multiagent coordination, consensus, and ?ocking[C]. *in Proc. 44th IEEE Conf. Decision Control Eur. Control Conf.*, Seville, Spain, Dec. 2005, pp. 2996 -C 3000. 1

[21] D. P. Bertsekas and J. N. Tsitsiklis. Comments on coordination of groups of mobile autonomous agents using nearest neighbor rules[J]. *IEEE Transactions on Automatic Control*, 2007, 52(5): 968 C- 969. 1

[22] L. Moreau and S. G. Belgium. Stability of continuous-time distributed consensus algorithms[C]. *in Proc. 43rd IEEE Conf. Decision Control*, Atlantis, Paradise Island, Bahamas, Dec. 2004, pp. 3998C4003. 1

[23] Sandro Zampieri. Trends in networked control systems[C], *Proceedings of 17th World Congress of IFAC*, Seoul, Korea, July 6-11, 2008, pp. 2886 – 2894. 2

[24] J.P. Desai, J.P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust[J]. *IEEE Int. Trans. on Robotics and Automation*, 2001, 17(6): 905 – 908. 2

[25] W. Ren, R.W. Beard. Formation feedback control for multiple spacecraft via virtual structures[J]. *IEE Proc. Control Theory and Applications*, 2004, 151(3): 357 – 368. 2

[26] H. Yamaguchi, T. Arai and G. Beni. A distributed control scheme for multiple robotic vehicles to make group formations[J]. *Robitics and Autonomous Systems*, 2001, 36: 125-147. 2

[27] Z.P. Wu, Z.H. Guan, T. Li. Group motion control of multi-agent systems based on complex network[C], *Proceedings of the 26th Chinese Control Conference*, Hunan, China, July 26-31, 2007, pp: 63 – 67. 2, 4

[28] M. Porfiri, D.G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach[J]. *Automatica*, 2007, 43: 1318 – 1328. 2

[29] J. Cortés. Global formation-shape stabilization of relative sensing networks[C]. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 1460 – 1465. 2

[30] Y. Liu, Y.M. Jia, J.P. Du and S.Y. Yuan. Dynamic output feedback control for consensus of multi-agent systems: an H_∞ approach[C]. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 4470 – 4475. 2

[31] Y.G. Hong, J.P. Hu and L.X. Gao. Tracking control for multi-agent consensus with an active leader and variable topology[J]. *Automatica*, 2006, 42: 1177 – 1182. 2

[32] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays[J]. *IEEE Transactions on Automatic Control*, 2008, 53(8): 1804 – 1816. 2

[33] T. Li and J.F. Zhang. Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash equilibria[J]. *Automatica*, 2008, 44: 713 – 725. 2

[34] E.S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information[J]. *Automatica*, 2008, 44: 2766 – 2777. 2

[35] R. Olfati-Saber and R.M. Murray. Consensus problmes in networks with switching topology and time-delays[J]. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1520 – 1533. 3

[36] M. Arack. Passivity as a design tool for group coordination[J]. *IEEE Transactions on Automatic Control*, 2007, 52(8): 1380 – 1390. 3

[37] F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu and Z.Z. Yuan. Reaching a consensus via pinning control[J]. *Automatica*, 2009, 45: 1215 – 1220. 3

[38] Y.P. Tian and C.L Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations[J]. *Automatica*, 2009, 45: 1347 – 1353. 3

[39] P. Lin, Y.M. Jia and L. Li. Distributed robust H_∞ Consensus control in directed networks of agents with time-delay[J]. *Systems and Control Letters*, 2008, 57(8): 643 – 653. 3

[40] P.A. Bliman and G.F. Trecate. Average consensus problems in networks of agents with delayed communications[J]. *Automatica*, 2008, 44: 1985 – 1995. 4

[41] J. Cortes. Distributed algorithms for reaching consensus on general functions[J]. *Automatica*, 2008, 44(3): 726 – 737. 4

[42] J. Liang, Z. Wang, X. Liu and P. Louvieris. Robust synchronization for two-dimensional discrete-time coupled dynamical networks[J]. *IEEE Transactions on Neural Networks and Learning Systems*, 2012, 23(6): 942 – 953. 4

[43] D. Ding, Z. Wang, B. Shen and H. Shu. H-infinity state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delays[J]. *IEEE Transactions on Neural Networks and Learning Systems*, 2012, 23(5): 725 – 736. 4

[44] B. Shen, Z. Wang and X. Liu. Bounded H-infinity synchronization and state estimation for discrete time-varying stochastic complex networks over a finite-horizon[J]. *IEEE Transactions on Neural Networks*, 2011, 22(1): 145 – 157. 4

[45] Z. Wang, Y. Wang and Y. Liu. Global synchronization for discrete-time stochastic complex networks with randomly occurred nonlinearities and

mixed time-delays[J]. *IEEE Transactions on Neural Networks*, 2010, 21(1): 11 – 25. 4

[46] Z. Ji, Z. Wang, H. Lin and Z. Wang. Interconnection topologies for multi-agent coordination under leader-follower framework[J]. *Automatica*, 2009, 45(12): 2857 – 2863. 4

[47] J. Liang, Z. Wang and X. Liu. State estimation for coupled uncertain stochastic networks with missing measurements and time-varying delays: the discrete-time case[J]. *IEEE Transactions on Neural Networks*, 2009, 20(5): 781 – 793. 4

[48] Y. Wang, Z. Wang and J. Liang. Global synchronization for delayed complex networks with randomly occurred nonlinearities and multiple stochastic disturbances[J]. *Journal of Physics A: Mathematical and Theoretical*, 2009, 42(13), art. No. 135101. 4

[49] Y. Liu, Z. Wang, J. Liang and X. Liu. Synchronization and state estimation for discrete-time complex networks with distributed delays[J]. *IEEE Transactions on Systems, Man, and Cybernetics - Part B*, 2008, 38(5): 1314 – 1325. 4

[50] G.P. Liu. Predictive controller design of networked systems with communication delays and data loss[J]. *IEEE Transactions on Circuits and Systems II*, 2010, 57(6): 481 – 485. 4

[51] C.Q. Ma and J.F. Zhang. Necessary and Sufficient Conditions for Consensusability of Linear Multi-Agent Systems[J]. *IEEE Transactions on Automatic Control*, 2010, 55(5): 1263 – 1268. 4

[52] A. Ajorlou, A. Momeni and A.G. Aghdam. A Class of Bounded Distributed

Control Strategies for Connectivity Preservation in Multi-Agent Systems[J]. *IEEE Transactions on Automatic Control*, 2010, 55(12): 2828 – 2833. 4

[53] Z.K. Li, Z.S. Duan, G.R. Chen and L. Huang. Consensus of multi-agent systems and synchronization of complex networks: a unified viewpoint[J]. *IEEE Transactions on Circuits and Systems-I*, 2009. 4

[54] A.L. Barabási, R. Albert, H. Jeong and G. Bianconi. Power-law distribution of the world wide web[J]. *Science*, 2000, 287: 2115a. 5

[55] S.H. Strogatz. Exploring complex network[J]. *Nature*, 2001, 410: 268 – 276. 5

[56] C.W. Wu. Synchronization in an array of linearly coupled dynamical systems[J]. *IEEE Trans.Circuits Syst. I*, 1995, 42: 430 – 447. 5

[57] A. Pogromsky, H. Nijmeijer. Cooperative oscillatory behavior of mutually coupled dynamical systems[J]. *IEEE Trans. Circuits Syst. I*, 2001, 48: 152 – 162. 5

[58] C.G. Li and G.R. Chen. Synchronization in general complex dynamical networks with coupling delays[J]. *Phys. A*, 2004, 343: 263 – 278. 5

[59] J.D. Cao, P. Li and W.W. Wang. Global synchronization in arrays of delayded neural networks with constant and delayde coupling[J]. *Phys. Lett. A*, 2006, 353: 318 – 325. 5

[60] Igor Belykh, Vladimir Belykh, Martin Hasler. Synchronization in complex networks with blingking interactions[C], *Proc. IEEE PhysCon.*Russia: St. Petersburg, 2005, 86 – 91. 5

[61] M. Dogruel, U. Ozgunzer. Stability of hybrid systems[C], *Proceedings of IEEE International Symposium on Intelligent Control*. New York: IEEE Press, 1994, 8: 129 – 134. 5

[62] Z.X. Liu, Z.Q. Chen and Z.Z. Yuan. Pinning control of weighted general complex dynamical networks with time delay[J]. *Phys. A*, 2007, 375: 345 – 354. 5

[63] C.P. Li, W.G. Sun and J.Kurths. Synchronization of complex dynamical networks with time delays[J] *Phys. A*, 2006, 361: 24 – 34. 5

[64] J.Q. Lu and Daniel W.C. Ho. Local and global synchronization in general complex dynamical networks with delay coupling[J]. *Chaos, Solitons Fractals*, doi:10.1016/j.chaos.2006.10.030. 5

[65] P. Li, Z. Yi, L. Zhag. Global synchronization of a class of delayed complex networks[J]. *Chaos, Solitons Fractals*, 2006, 30: 903 – 908. 5

[66] W.J. Xiong, W. Xie and J.D. Cao. Adaptive exponential synchronization of delayde chaotic networks[J]. *Phys. A*, 2006, 370: 832 – 842. 5

[67] V.N. Belykh, I. Belykh, M. Hasler. Connection graph stability method for synchronized coupled chaotic systems[J]. *Phys. D*, 2004, 195: 159 – 187. 5

[68] X.F. Wang, G.R. Chen. Synchronization in scale-free dynamical networks: robustness and fragility[J]. *IEEE Tran.Circuits and Syst.I*, 2002, 49: 54 – 62. 5

[69] X. Li, G.R. Chen and King-Tim Ko. Transition to chaos in complex dynamical networks[J]. *Phys. A*, 2004, 338: 367 – 378. 5

[70] Z. Li and G.R. Chen. Global synchronization and asymptotic stability of complex dynamical networks[J]. *IEEE Tran.Circuits and Syst.I*, 2006, 53: 28 – 33. 5

[71] X. Li and G.R. Chen. Synchronization and desynchronization of complex dynamical networks: an engineering viewpoint[J]. *IEEE Tran.Circuits and Syst.I*, 2003, 50: 1381 – 1390. 5

[72] X.F. Wang and G.R. Chen. Complex networks: small-world, scale-free and beyond[J]. *IEEE Circuits and Syst. Mag.*, 2003, 3: 6-20. 5

[73] J.H. Lü, X.H. Yu, G.R. Chen and D.Z. Chen. Characterizing the synchronizability of small-world dynamical networks[J]. *IEEE Tran.Circuits and Syst.I*, 2004, 51: 787 – 796. 5

[74] X.F. Wang and G.R. Chen. Pinning control of scale-free dynamical networks[J]. *Phys. A*, 2002, 310: 521 – 531. 5

[75] X. Li, X.F. Wang and G.R. Chen. Pinning a complex dynamical network to its equilibrium[J] *IEEE Tran.Circuits and Syst.I*, 2004, 51: 2074 – 2087. 5

[76] J.H. Lü, G.R. Chen. A time-varying Complex Dynamical network model and its controlled synchronization criteria[J]. *IEEE Trans. Auto. Contr.*, 2005, 50: 841 – 846. 5

[77] Z. Li, G.R. Chen. Robust adaptive synchronization of uncertain dynamical networks[J]. *Phys.Lett.A*, 2004, 324: 166 – 178. 5

[78] J. Zhou, J.A Lu and J.H. Lü. Adaptive synchronization of an uncertain complex dynamical network[J]. *IEEE Trans. Auto. Contr.*, 2006, 51: 652 – 656. 5

[79] C.H. Cai and G.R. Chen. Synchronization of complex dynamical networks by the incremental ISS approach[J]. *Phys. A*, 2006, 371: 754 – 766, . 5

[80] W.W. Wang, J.D. Cao. Synchronization in an array of linearly coupled networks with wime-varying delay[J]. *Phys. A*, 2006, vol. 366: 197 – 211. 5

[81] M.Y. Chen. Some simple synchronzation criteria for complex dynamical networks[J]. *IEEE Tran.Circuits and Syst.I*, 2006, 53: 1185 – 1189. 5

[82] G.P. Jiang, W.K.S. Tang and G.R. Chen. A state-observer-based approach for synchronization in complex dynamical networks[J]. *IEEE Tran.Circuits and Syst.I*, 2006, 53: 2739 – 2745. 5

[83] W.L. Lu, T.P. Chen and G.R. Chen. Synchronization analysis of linearly coupled systems described by differential equations with a coupling delay. *Physica D*, 2006, 221(2): 118 – 134. 5

[84] J.Q. Lu, D.W.C. Ho and J. Kurths. Consensus over directed static networks with arbitrary finite communication delays. *Physical Review E*, 2009, 80(6): 066121. 5

[85] J.Q. Lu and D.W.C. Ho. Globally Exponential Synchronization and Synchronizability for General Dynamical Networks. *IEEE Transactions on Systems, Man and Cybernetics, Part B*, 2010, 40(2): 350 – 361. 5

[86] M. Brede. Synchrony-optimized networks of non-identical Kuramoto oscillators[J]. *Physics Letters A*, 2008, 372(15): 2618 – 2622. 5

[87] J. Xiang and G.R. Chen. On the V-stability of complex dynamical networks[J]. *Automatica*, 2007, 43: 1049 – 1057. 5

[88] D. J.Hill and J. Zhao. Global synchronization of complex dynamical networks with non-identical nodes[C], *Proceedings of the 47th IEEE Conference on*

Decision and Control, Cancun, Mexico, Dec. 9-11, 2008, 817 – 822. 5, 47, 48, 70, 82, 83, 102, 114, 115, 119, 136, 149, 150

[89] J. Zhao, D. J. Hill and T. Liu. Synchronization of Dynamical Networks With Nonidentical Nodes: Criteria and Control. *IEEE Tran.Circuits and Syst.I*, 2011, 58(3): 584 – 594. 5, 136

[90] U.E. Vincent and J.A. Laoye. Synchronization, anti-synchronization and current transports in non-identical chaotic ratchets[J]. *Physica A*, 2007, 384: 230 – 240. 5

[91] R.K. Upadhyay, V. Rai. Complex dynamics and synchronization in two non-identical chaotic ecological systems[J]. *Chaos, Solutions and Fractals*, 2009, 40(5): 2233 – 2241. 5

[92] C.C. Hua, X.P. Guan and P. Shi. Decentralized robust model reference adaptive cotnrol for interconnected time-delay systems[J]. *J.Comput.Appl.Math.*, 2006, 193: 383 – 396. 33, 47, 86

[93] X.X. Liao. *Theory methods and application of stability*[M]. China: HUST Press, 1999. 47

[94] Y.M. Moon, P. Park, W.H. Kwon, Y.S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems[J]. *Int J. Control*, 2001, 74: 1447 – 1455. 33, 69, 85

[95] Pin-Lin Liu. Exponential stability for linear time-delay systems with delay dependence[J]. *Journal of the Franklin Institute*, 2003, 340: 481 – 488. 69

[96] X.M. Sun, J. Zhao and D.J. Hill. Stability and L_2-gain analysis for switched delay systems: A delay-dependent method[J]. *Automatica*, 2006, 42: 1769 – 1774. 69

[97] J.K. Hale, S.M.V. Lunel. *Introduction to funtional differential equations*[M]. German: Springer-Verlag, 1993. 69

[98] H. Chen. *Stochastic Approximation and Its Applications*[M]. Dordrecht: Kluwer Academic Publisher, 2002. 69

[99] V.B. Kolmanovskii and A.D. Myshkis, *Introduction to the theory and applications of funtional differential equations.* Dordrecht: Kluwer Academic Publishers, 1999. 69

[100] Chien-Hsin Chou, Chih-Chiang Cheng. A dencentralized model reference adaptive variable structure controller for large-sacle time-varying delay systems[J]. *IEEE Trans. Auto. Contr.*, 2003, vol. 48: 1313 – 1217. 69

Publications & Finished Papers

[1] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Consensus of Multi-Agent Systems with Non-Identical Nodes and Time Delays. *IEEE Transactions on Systems, Man, and Cybernetics-Part B*. Available online.

[2] W.S. Zhong, G.P. Liu and D. Rees. Global Bounded Consensus of Multi-Agent Systems with Non-Identical Nodes and Communication Time-Delay Topology. *International Journal of Systems Science*, DOI:10.1080/00207721.2011.601346. Available online.

[3] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus of Multi-Agent Systems with Communication Time Delay. *In the 2nd International Conference on Intelligent Control and Information Processing*, pages 775-778, Harbin, China, 25-28 July, 2011.

[4] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus Criteria for a Class of Uncertain Multi-Agent Systems. *In the 18th IFAC World Congress*, pages 5448-5452, Milano, Italy, 28 August- 2 September, 2011.

[5] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus of Multi-Agent Systems with Different Agent Dynamics. *In the UKACC International Conference on Control 2010*, pages 542-547, Coventry, UK, 7-10 September, 2010.

[6] W.S. Zhong, G.P. Liu, D. Rees and C. Thomas. Global Bounded Controlled Consensus of Networked Multi-Agent Systems with Non-Identical Dynamical Agents. *In the 16th International Conference on Automation & Computing*, pages 186-191, Birmingham, UK, 11 September, 2010.

[7] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Controlled Consensus of Networked Multi-Agents Systems with Non-Identical Dynamical Agents. *In the 18th IFAC World Congress*, pages 245-250, Milano, Italy, 28 August- 2 September, 2011.

[8] W.S. Zhong, G.P. Liu and C. Thomas. Global Consensus Analysis of Networked Multi-Agent Systems with Heterogeneous Dynamics and Time-Varying Communication Delay. *In the 8th Japan-China International Workshop on Internet Technology and Control Applications.* Tokyo, Japan, 5-11 December, 2011.

[9] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Controlled Consensus of Multi-Agents Systems with Non-Identical Nodes and Communication Time-Delay Topology. *In the UKACC International Conference on Control 2012*, Cardiff, UK, 3-5 September, 2012. Accepted

[10] W.S. Zhong, G.P. Liu and C. Thomas. Global Controlled Consensus of Multi-Agent Systems with Different Agent Dynamics and Time-Varying Communication Delay. *In the UKACC International Conference on Control 2012*, Cardiff, UK, 3-5 September, 2012. Accepted

[11] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Controlled Consensus

of Multi-Agents Systems with Non-Identical Nodes and Communication Time-Delay Topology. *IEEE Transactions on Circuits and Systems-II*. Submitted.

[12] W.S. Zhong, G.P. Liu and C. Thomas. Global Bounded Controlled Consensus of Delayed Multi-Agents Systems with Non-Identical Nodes. *IEEE Transactions on Automatic Control*. Submitted.

[13] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Global Exact Consensus of Multi-Agent Systems with Communication Time Delay. *International Journal of Robust and Nonlinear Control*. Submitted.

[14] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus Criteria for NMAS with Uncertain Coupling. *IET Control Theory & Applications*. Submitted.

[15] W.S. Zhong, G.P. Liu and C. Thomas. Controlled Consensus of Multi-Agent Systems with Non-Identical Agent Dynamics. *IET Control Theory & Applications*. Submitted.

Global Bounded Consensus of Multiagent Systems With Nonidentical Nodes and Time Delays

Wei-Song Zhong, Guo-Ping Liu, *Fellow, IEEE*, and Clive Thomas

***Abstract*—This paper investigates the global bounded consensus problem of networked multiagent systems consisting of nonlinear nonidentical node dynamics with the communication time-delay topology. We derive globally bounded controlled consensus conditions for both delay-independent and delay-dependent conditions based on the Lyapunov–Krasovskii functional method. The proposed consensus criteria ensure that all agents eventually move along the desired trajectory in the sense of boundedness. Meanwhile, the bounded consensus criteria can be viewed as an extension of the case of identical agent dynamics to the case of nonidentical agent dynamics. We finally demonstrate the effectiveness of the theoretical results by means of a numerical simulation.**

***Index Terms*—Complex dynamical network, consensus, multiagent systems (MASs), networked control systems.**

I. Introduction

MULTIAGENT SYSTEMS (MASs) analysis involves the study of how network architectures and interactions between network components influence global control goals. The research in this field can be categorized into two areas: One is to deal with the design of distributed estimation techniques which can be applied to the sensor networks, and the other is to deal with the control of mobile autonomous agents where each agent acts autonomously using information obtained over the network from other neighboring agents [1]. In both areas, some important contributions have been made in recent years [2]–[9].

The consensus problem requires an agreement to be reached that depends on the states of all agents, i.e., to design distributed control strategies based on local information that enables all agents to reach some kind of agreements on certain quantities of interest. The topic has been studied across many fields of science and engineering, and many results have been achieved [10]–[21]. In the context of MASs, many pioneering contributions involved with various distributed strategies that achieve consensus have been witnessed. Olfati-Sabre introduced two consensus criteria for networks with and without time delays and provides a convergence analysis for three kinds of MAS with fixed and switching topologies [22]. A passivity-based design framework developed to process the group coordination problem, with both fixed and time-varying communication structures, has also been considered [23]. All agents reach a consensus if a small fraction of them are controlled by simple feedback control, which is demonstrated in [24]. The robust consensus problems of second-order MAS with diverse input delays are investigated, and decentralized consensus conditions are obtained for the MAS with symmetric coupling weights based on the frequency-domain analysis in [25]. The consensus problem for directed MAS with external disturbances and model uncertainties for fixed and switching topologies is discussed in [26]. The average consensus problem for undirected MAS having communication delays is studied, and sufficient conditions are provided for the existence of average consensus under bounded communication delays in [27]. A distributed algorithm that asymptotically achieved consensus is characterized, and two discontinuous distributed algorithms that achieve maximum and minimum consensus, respectively, are provided in [28]. It should be noted that the agent dynamics involved in most existing results are often restricted to be linear and identical ones. However, this is not always the case in practice, and significant differences exist widely within the relevant agents. Strictly speaking, each agent, regardless of the similarity in its main functions, has characteristics that exhibit a degree of difference, particularly in its isolated agent dynamics which are usually modeled as nonlinear dynamical systems.

The consensus analysis of a MAS consisting of nonidentical agent dynamics is much more complicated than that of the identical case, and few results have been reported to date. However, the following idea widely used in the complex dynamical network can be applied to deal with the consensus analysis of the MAS. The similarity between the consensus of MAS and the synchronization of complex dynamical networks suggests a way forward [5], [29]. A complex network is a large set of interconnected dynamic nodes where its specific representation is determined by the specific application. It has attracted tremendous attention in recent years [30], [31]. Since the connection topology plays a key role in forming the behaviors of a complex network, researchers have examined a variety of connection topologies and tried to better understand how topology influences the network behavior. Synchronization is one of the key issues that affect network behavior and has been extensively addressed, and a large number of papers on this topic have appeared based on complex networks with identical nodes [32]–[42]. As for the synchronization of

Manuscript received February 7, 2011; revised October 7, 2011 and January 24, 2012; accepted February 24, 2012. The work of G.-P. Liu was supported in part by the National Natural Science Foundation of China under Grant 61028010 and Grant 61021002. This paper was recommended by Associate Editor T. H. Lee.

W.-S. Zhong and C. Thomas are with the Faculty of Advanced Technology, University of Glamorgan, CF37 1DL Pontypridd, U.K. (e-mail: wzhong@glam.ac.uk; ejcthomas@glam.ac.uk).

G.-P. Liu is with the Faculty of Advanced Technology, University of Glamorgan, CF37 1DL Pontypridd, U.K., and also with the Center for Control Theory and Guidance Technology Center, Harbin Institute of Technology, Harbin 150001, China (e-mail: gpliu@glam.ac.uk).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSMCB.2012.2192428

complex networks with nonidentical nodes, some results have been proposed. A simulation-based synchronization study for nonidentical Kuramoto oscillators was carried out in [43]. A simple case where all nonidentical nodes have the same equilibrium was considered in [44], and a synchronization criterion was given by constructing the same Lyapunov function for all the nodes. [45] and [46] studied the synchronization problem for a complex dynamical network with nonidentical nodes, and the proposed results extend the relevant asymptotic synchronization criteria to this case. Several collective properties for coupled nonidentical chaotic systems were discussed in [47] and [48]. Therefore, if the ideas in the synchronization problem of complex dynamical networks are applied properly, then consensus problems are solvable.

Inspired by preliminary results [49]–[51], this paper will focus on the global consensus problems of MAS, and the proposed consensus property is formulated in terms of the certain boundedness of state errors which can be interpreted as the difference between individuals. The reason why the behavior of the MAS with nonidentical agent dynamics is much more complicated than that of the identical case are summarized as follows. Usually, no common equilibrium for all agents exists even if each agent has an equilibrium; neither does a consensus manifold exist in the classical sense. The consensus of a MAS with identical agents is usually described in terms of the (asymptotically) identical dynamical evolution of state variables of every agent in the MAS, which is easy to understand. However, this collective behavior, called exact consensus, no longer exists in the MAS with nonidentical agents due to the difference between the dynamics of the agents. Furthermore, the MAS with nonidentical agent dynamics cannot be decomposed into a number of lower dimensional systems exactly like the identical-agent case. However, a MAS with nonidentical agents may still exhibit some kinds of consensus behaviors which are far from being fully understood, and very few results have been reported to date. Certain reasonable and satisfactory boundedness of state motion errors between different agents can be taken as useful consensus properties. Compared with many existing results, this paper makes several significant contributions. First, we generalize the related results for the case of identical agent dynamics to the case of nonidentical agent dynamics, and the proposed results cover the existing criteria of networks with identical agent dynamics as special cases. Second, we consider the communication time delay among the agents; global consensus criteria are given based on solving a number of lower dimensional matrix inequalities and scalar inequalities, which generalize the criteria using the method of the master stability function for MAS with identical agents. Finally, globally bounded consensus conditions for both delay-independent and delay-dependent conditions based on the Lyapunov–Krasovskii functional method are derived.

The rest of this paper is organized as follows. A continuous-time MAS model with nonidentical agent dynamics and communication time delays is presented, and some preliminaries are introduced in Section II. The main results including delay-independent and delay-dependent bounded consensus criteria are derived in Section III. In Section IV, a numerical-simulation example is given to verify the effectiveness of the proposed results, followed by conclusions in Section V.

II. Preliminaries

A. Problem Description

Let $G = (\mathcal{V}, \mathcal{A})$ be a graph of order N which consists of a set of vertices $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and a set of edges $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$. An edge (v_j, v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{A}\}$.

We consider a MAS consisting of N nonidentical agents with communication delay

$$\dot{x}_i(t) = f_i\left(x_i(t)\right) + u_i, \qquad i = 1, 2, \ldots, N \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i; $f_i(x_i(t)) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i; $u_i = c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma(x_{j1}(t - \tau_{j1}), x_{j2}(t - \tau_{j2}), \ldots, x_{jn}(t - \tau_{jn}))^T$; $c > 0$ denotes the coupling strength; $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix; and if $\gamma_{ij} \neq 0$, then it means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N\times N}$, which is symmetric and irreducible, representing the communication-topology relation of the MAS, is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$, and $a_{ii} = -\sum_{j\neq i} a_{ij}$. The time-delay vector $(\tau_{j1}, \tau_{j2}, \ldots, \tau_{jn})$ reflects the reality that the agent v_i cannot obtain information from agent v_j instantaneously.

Remark 2.1: The consensus problem of MAS is usually viewed as the synchronization of coupled nonlinear oscillators, and there are no significant differences between the MAS and the complex dynamical networks. The synchronization problem of the complex dynamical network usually emphasizes its nonlinear node dynamics; thus, only sufficient conditions can be given for verifying the synchronization. However, in the context of MASs, researches are mainly focusing on designing various distributed strategies which can guarantee the MAS to achieve agreements. In many cases, the agent dynamics are usually restricted to be single or double integrators or high-order linear systems, and the most proposed distributed consensus protocols are usually based on the relative states between neighboring agents. However, the fact is that many necessary and sufficient conditions can be given.

The average dynamics of all agents is defined by the vector field

$$\bar{f}\left(x(t)\right) = \frac{1}{N}\sum_{k=1}^{N} f_k\left(x(t)\right). \tag{2}$$

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t). \tag{3}$$

The consensus-problem formulation in this paper is quite different from many others, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)\ \forall i, j = 1, 2, \ldots, N$ as $t \to \infty$, i.e., there exists a function $s(t)$ such that $\lim_{t\to\infty}(x_i(t) - s(t)) = 0$. The consensus problem here will be depicted instead via a certain boundedness of $x_i(t) - x_j(t)$ $\forall i, j = 1, 2, \ldots, N$ as $t \to \infty$. This better reflects reality when it is impossible for MAS (1) to achieve exact consensus. To address this case, we will focus on making the states of all agents converge to a bounded set.

B. Mathematical Preliminaries

Before stating the main results of this paper, the following mathematical preliminaries are necessary.

Definition 1 [52]: The solution $x_i(t, t_0, \psi_i)$ of the MAS (1) is said to be uniformly ultimately bounded with respect to the bound ε if for each $\delta > 0$, there exists $T = T(\varepsilon, \delta) > 0$ independent of t_0 such that $\|x_i(t, t_0, \psi_i)\| \le \varepsilon$ for all $t \ge t_0 + T$ when $\|x_i(t_0)\| < \delta$, where ψ_i is the initial value given as $x_i(t) = \psi_i$ for $t \in [t_0 - \tau, t_0]$, $i = 1, 2, \ldots, N$.

Lemma 1 [45]: Let $g(t)$ be a nonnegative bounded function defined on R^+ and

$$\Omega = \left\{x(t) \in R^n \middle| \|x(t)\| \le \overline{\lim}_{t\to\infty} g(t)\right\}. \tag{4}$$

Suppose there exist a strictly positive definite matrix $P(t) \in \mathcal{PC}^1_{n\times n}$ and a constant $\delta > 0$ such that the derivative of $V(x(t), t) = x^T(t)P(t)x(t)$ along the trajectory of the system

$$\dot{x}(t) = f\left(x(t), t\right), \qquad x(t) \in R^n; \quad t \in [0, \infty) \tag{5}$$

satisfies

$$\dot{V} \le -\delta \|x(t)\|^2 \qquad \texttt{if} \quad \|x(t)\| \ge g(t). \tag{6}$$

For any $t > 0$, let

$$Q_t = \left\{x(t) | V\left(x(t), t\right) \le \sup_{y(s)\in\Omega, s\ge 0} \left\{V\left(y(s), s\right)\right\}\right\} \tag{7}$$

$$\bar{c} = \overline{\lim}_{t\to\infty}\left(\max\left\{\|x(t)\| \,|x(t) \in Q_t\right\}\right). \tag{8}$$

Then, $x(t)$ converges to the set

$$M = \{x(t) | \, \|x(t)\| \le \bar{c}\}. \tag{9}$$

Lemma 2 [53]: Assume that $a(.) \in R^{n_a}$, $b(.) \in R^{n_b}$, and $M(.) \in R^{n_a\times n_b}$ are defined on an interval Ω. Then, for any matrices $X \in R^{n_a\times n_a}$, $Y \in R^{n_a\times n_b}$, and $Z \in R^{n_b\times n_b}$, the following inequality holds:

$$-2\int_\Omega a^T(\alpha)Mb(\alpha)d\alpha \le \int_\Omega \begin{pmatrix} a(\alpha) \\ b(\alpha)\end{pmatrix}^T \Delta \begin{pmatrix} a(\alpha) \\ b(\alpha)\end{pmatrix} d\alpha$$

$$\Delta = \begin{pmatrix} X & Y - M \\ Y^T - M^T & Z \end{pmatrix}, \qquad \begin{pmatrix} X & Y \\ Y^T & Z\end{pmatrix} \ge 0. \tag{10}$$

We now introduce some notations and definitions.

Let $\overline{x_j(t-\tau)} = (x_{j1}(t-\tau_1), x_{j2}(t-\tau_2), \ldots, x_{jn}(t-\tau_n))^T$ and $\overline{x_j(t-\tau_j)} = (x_{j1}(t-\tau_{j1}), x_{j2}(t-\tau_{j2}), \ldots, x_{jn}(t-\tau_{jn}))^T$.

Let $PC^1_{n\times n}$ be the linear space of the uniformly bounded continuous real matrix-valued functions defined on $[0, \infty)$. For any $P \in PC_{n\times n}$, the norm of P is defined by $\|P\| = \max_{0\le t<\infty}\{\|P(t)\|\}$.

Let "$\otimes$" be the Kronecker product.

Let $\mu_2(A)$ be half the maximum eigenvalue of $\bar{A}^T + A$.

III. Main Results

Define the error vector as

$$e_i(t) = x_i(t) - s(t), \qquad i = 1, 2, \ldots, N. \tag{11}$$

Obviously, $\sum_{i=i}^N e_i = 0$, and $(c/N)\sum_{k=1}^N \sum_{j\in\mathcal{N}_i}^N a_{kj}\Gamma\overline{x_j(t-\tau_j)} = 0$; then, the MAS (1) can be rewritten in terms of e_i as

$$\dot{e}_i(t) = f_i\left(s(t) + e_i(t)\right) - \frac{1}{N}\sum_{k=1}^N f_k\left(s(t) + e_k(t)\right) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma\overline{e_j(t-\tau_j)} \tag{12}$$

where $\overline{e_j(t-\tau_j)} = (e_{j1}(t-\tau_{j1}), e_{j2}(t-\tau_{j2}), \ldots, e_{jn}(t-\tau_{jn}))^T$.

The following work will focus on simplifying the error MAS (12) by means of a series of transformations, and the procedure is similar to [45] and [46].

According to the Newton–Leibniz formula, the error MAS (12) can be written further as

$$\begin{aligned}\dot{e}_i(t) &= D\bar{f}\left(s(t)\right)e_i(t) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma\overline{e_j(t-\tau_j)} \\ &+ \int_0^1 \left(Df_i\left(s(t) + \tau e_i(t)\right) - D\bar{f}\left(s(t)\right)\right)e_i(t)d\tau \\ &- \frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k\left(s(t) + \tau e_k(t)\right)e_k(t)d\tau \\ &+ f_i\left(s(t)\right) - \bar{f}\left(s(t)\right)\end{aligned} \tag{13}$$

where $D\bar{f}(s(t))$ is the Jacobian matrix of $\bar{f}(x(t))$ at $s(t)$ and $\bar{f}(x(t))$ is defined by (refaveragedynamic).

If we consider the linearized MAS of (12), we have

$$\begin{aligned}\dot{e}_i(t) &= D\bar{f}\left(s(t)\right)e_i(t) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma\overline{e_j(t-\tau_j)} \\ &+ \left(Df_i\left(s(t)\right) - D\bar{f}\left(s(t)\right)\right)e_i(t) \\ &- \frac{1}{N}\sum_{k=1}^N Df_k\left(s(t)\right)e_k(t) + f_i\left(s(t)\right) - \bar{f}\left(s(t)\right).\end{aligned} \tag{14}$$

Let $e(t) = \left(e_1^T(t), e_2^T(t), \ldots, e_N^T(t)\right)^T$; then, (13) becomes

$$\dot{e}(t) = I_N \otimes D\bar{f}(s)e(t) + cA \otimes \Gamma\overline{e(t-\tau_j)} + I(t)e(t) - \frac{1}{N}H(t)e(t) + F(t) \tag{15}$$

$I(t) = \mathrm{diag}\{\int_0^1(Df_1(s + \tau e_1) - D\bar{f}(s))d\tau \ldots \int_0^1(Df_N(s + \tau e_N) - D\bar{f}(s))d\tau\}$, $\bar{\Sigma}(t) = I_N \otimes D\bar{f}(s) + D$, $H^T(t) = (H_1^T(t),$

$\ldots, H_N^T(t))$, $H_i(t) = (\int_0^1 Df_1(s+\tau e_1)d\tau, \ldots, \int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t) = (f_1^T(s) - \bar{f}^T(s), \ldots, f_N^T(s) - \bar{f}^T(s))$.

Since the graph $G = (\mathcal{V}, \mathcal{A})$ is a strongly connected graph and A is symmetric and irreducible, there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \ldots, \Phi_N)$, such that the adjacency matrix A satisfies $\Phi^T A \Phi = \Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, where Φ_i is the ith column of Φ with $\Phi_1 = (1/\sqrt{N}, 1/\sqrt{N}, \ldots, 1/\sqrt{N})^T$ and $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$ as the eigenvalues of A. Let $\omega(t) = (\Phi^T \otimes I_n)e(t)$; then

$$\begin{aligned}\dot{\omega}(t) &= (\Phi^T \otimes I_n)\dot{e}(t)\\ &= (\Phi^T \otimes I_n)\left(I_N \otimes D\bar{f}(s)\right)(\Phi \otimes I_n)\omega(t)\\ &\quad + (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)\overline{\omega(t-\tau_j)}\\ &\quad + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)\\ &\quad - \frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)\omega(t)\\ &\quad + (\Phi^T \otimes I_n)F(t). \end{aligned} \tag{16}$$

Note that $H(t) = \sqrt{N}\sum_{k=1}^N (\mathbf{0}\cdots\mathbf{0}\bar{\Phi}_k\mathbf{0}\cdots\mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its kth column equal to Φ_1 and the remaining elements are zero; then, we have $(1/N)(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = (1/\sqrt{N})\sum_{k=1}^N (\mathbf{0}\cdots\mathbf{0}I_k\mathbf{0}\cdots\mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau(\Phi \otimes I_n)$, where I_k stands for the matrix with its kth column equal to $(1\ 0\ \cdots\ 0)^T$ and the remaining elements are zero. Thus, a simple calculation gives $(1/N)(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = (1/\sqrt{N})\sum_{k=1}^N \binom{\Upsilon_k}{0} \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k \in R^{1\times N}$ and $0 \in R^{(N-1)\times N}$. Therefore, $\dot{\omega}(t) = I_N \otimes D\bar{f}(s)\omega(t) + c\Lambda \otimes \Gamma\overline{\omega(t-\tau_j)} + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) - \binom{*}{0}\omega(t) + (\Phi^T \otimes I_n)F(t)$. Since $w_1 \equiv 0$, we only need to consider $w_2, w_3, \ldots, w_N$. Rewriting in the component form, we have

$$\begin{aligned}\dot{\omega}_i(t) &= D\bar{f}(s(t)\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}\\ &\quad + \left(\Phi_i^T \otimes I_n\right)I(t)(\Phi \otimes I_n)\omega(t)\\ &\quad + \left(\Phi_i^T \otimes I_n\right)F(t), \qquad i = 2, 3, \ldots, N. \end{aligned} \tag{17}$$

In the following, a delay-independent global consensus criterion is derived for the MAS (1).

Theorem 3.1: Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$ and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$, and $b > 0$ such that

$$a\|x(t)\|^2 \leq x^T(t)P_i(t)x(t) \leq b\|x(t)\|^2 \quad \forall t \in R^+;\quad x \in R^n;\quad i = 2, 3, \ldots, N \tag{18}$$

$$\mu_2\left(\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma\right) + \zeta I \leq 0,\quad i = 1, 2, \ldots, N \tag{19}$$

$$\|I(t)\| \leq \gamma, \qquad i = 1, 2, \ldots, N \tag{20}$$

and there exist an i and a j such that

$$\begin{pmatrix} -\lambda_j(\Gamma + \Gamma^T) & \lambda_i\Gamma \\ \lambda_i\Gamma^T & 0 \end{pmatrix} \leq 0 \tag{21}$$

for all $t \geq t_0$, where $i = 2, 3, \ldots, n$ and $j = 1, 2, \ldots, n$.

Let

$$\mu(t) = \|F(t)\| \tag{22}$$

be bounded and

$$\beta = \left(\sum_{i=2}^N \|P_i(t)\|^2\right)^{\frac{1}{2}} \tag{23}$$

if $\zeta > 2\gamma\beta$. Then, the system (15) converges to the set

$$M = \left\{e(t) \middle| \|e(t)\| \leq \frac{2b}{a}\frac{\beta\overline{\lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\right\} \tag{24}$$

namely, $e(t) = x_i(t) - (1/N)\sum_{k=1}^N x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, and then, the MAS (1) achieves global bounded consensus for any fixed time delays $\tau_{kl} > 0 (k, l = 1, 2, \ldots, n)$.

Proof: Choose the following Lyapunov function as

$$V(w_i(t), t) = \sum_{i=2}^N V_i(w_i(t), t) \tag{25}$$

$$V_i(w_i(t), t) = w_i^T(t)P_i(t)w_i(t), \qquad i = 2, 3, \ldots, N. \tag{26}$$

Differentiating (26) along the trajectory of (17) gives

$$\begin{aligned}&\dot{V}_i(w_i(t), t)\\ &= w_i^T(t)\Big(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t))\\ &\qquad + D\bar{f}(s(t))^T P_i(t)\Big)w_i(t)\\ &\quad + 2w_i^T(t)P_i(t)\left(\Phi_i^T \otimes I_n\right)I(t)(\Phi_i \otimes I_n)w(t)\\ &\quad + 2w_i^T(t)P_i(t)\left(\Phi_i^T \otimes I_n\right)F(t)\\ &\quad + 2w_i^T(t)\left(c\lambda_iP_i(t)\Gamma\right)\overline{w_i(t-\tau_j)}\\ &= w_i^T(t)\Bigg(\left(\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma\right)\\ &\qquad + \left(\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma\right)^T\Bigg)w_i(t)\\ &\quad + c\begin{pmatrix} w_i(t) \\ w_i(t-\tau_j)\end{pmatrix}^T\Theta\begin{pmatrix} w_i(t) \\ w_i(t-\tau_j)\end{pmatrix}\\ &\quad + 2w_i^T(t)P_i(t)\left(\Phi_i^T \otimes I_n\right)I(t)(\Phi_i \otimes I_n)w(t)\\ &\quad + 2w_i^T(t)P_i(t)\left(\Phi_i^T \otimes I_n\right)F(t) \end{aligned} \tag{27}$$

where $\Theta = \begin{pmatrix} -\lambda_j(\Gamma + \Gamma^T) & \lambda_i\Gamma \\ \lambda_i\Gamma^T & 0 \end{pmatrix}$.

Condition (19) implies that the first term on the right-hand side of (27) satisfies

$$\begin{aligned}&w_i^T(t)\Bigg(\left(\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma\right)\\ &\qquad + \left(\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma\right)^T\Bigg)w_i(t)\\ &\quad \leq -\zeta\|w_i(t)\|^2. \end{aligned} \tag{28}$$

Applying condition (20), we know that the second term on the right-hand side of (27) satisfies

$$2w_i^T(t)P_i(t)\left(\Phi_i^T \otimes I_n\right) I(t)(\Phi_i \otimes I_n)w(t) \leq 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|. \quad (29)$$

The third term on the right-hand side of (27) satisfies

$$2w_i^T(t)P_i(t)\left(\Phi_i^T \otimes I_n\right) F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i(t)\|. \quad (30)$$

Since $V(w(t),t) = \sum_{i=2}^N V_i(w_i(t),t)$, we have

$$\begin{aligned}\dot{V}(w(t),t) &= \sum_{i=2}^N \dot{V}_i(w_i(t),t) \leq \sum_{i=2}^N \left(-\zeta\|w_i(t)\|^2\right)\\ &\quad + 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|\\ &\quad + 2\mu(t)\|P_i(t)\|\|w_i(t)\|\\ &= -\zeta\|w(t)\|^2\\ &\quad + 2(\gamma\|w(t)\| + \mu(t))\sum_{i=2}^N \|w_i(t)\|\|P_i(t)\|\\ &\leq -\zeta\|w(t)\|^2\\ &\quad + 2(\gamma\|w(t)\|+\mu(t))\|w(t)\|\left(\sum_{i=2}^N \|P_i(t)\|^2\right)^{\frac{1}{2}}\\ &= \|w(t)\|\left((2\gamma\beta - \zeta)\|w(t)\| + 2\beta\mu(t)\right). \end{aligned} \quad (31)$$

Thus, when

$$\|w(t)\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta} \quad (32)$$

we have

$$\dot{V}(w(t),t) \leq -\delta\|w(t)\|^2. \quad (33)$$

Applying Lemma 1 completes the proof.

Remark 3.1: The aforementioned result is a delay-independent globally consensus criterion, and the ultimate convergence bound is evaluated by means of (24). Theorem 3.1 guarantees that all agents move along the desired trajectory described by $s(t)$ in terms of certain boundedness, i.e., the consensus achieved here is just approximate instead of exact; in fact, to achieve exact consensus is impossible for such a case.

Remark 3.2: We have an asymptotic consensus criterion in the classical sense when $\overline{\lim}_{t\to\infty}\mu(t) = 0$. In particular, we have $\mu(t) \equiv 0$ when all agents have the same dynamics, i.e., $f_i(x_i(t)) = f(x(t))$. In such a case, applying Theorem 3.1 to the linearized network (14), which is equivalent to taking $\gamma = 0$ in (20), immediately achieves the universal consensus criteria existing in many literatures. Therefore, Theorem 3.1 covers the existing criteria of networks with identical agent dynamics as a special case.

Next, we will provide a delay-dependent bounded consensus criterion for the proposed problem.

Theorem 3.2: Suppose that (18) and (20) in Theorem 3.1 are satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Pi_i > 0$, $Z_i > 0$, $\Sigma_i > 0$, X_i, and Y_i of appropriate dimensions such that

$$\Xi = \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{pmatrix} < 0 \quad (34)$$

where

$$\begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0 \quad (35)$$

for $i = 2, 3, \ldots, N$, $\Xi_{11} = \dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + nhX_i + nQ_i + 2n(Y_i - (c\lambda_i/n)P_i(t)\Gamma) + nhD\bar{f}(s(t))^T Z_i D\bar{f}(s(t))$, $\Xi_{12} = (c\lambda_i/n)P_i(t)\Gamma - Y_i + c\lambda_i nhD\bar{f}(s(t))^T Z_i\Gamma$, and $\Xi_{22} = \Pi_i^{-1} + \Sigma_i^{-1} - Q_i + c^2\lambda_i^2 nh\Gamma^T Z_i\Gamma$, then the MAS (1) achieves global bounded consensus for any fixed time delays $\tau_{kl} \in [0,h] > 0 (k,l = 1,2,\ldots,n)$ for some $h < \infty$.

Proof: Construct the following Lyapunov–Krasovskii functional as

$$V(w_i(t),t) = \sum_{i=2}^N \sum_{k=1}^3 V_k(w_i(t),t) \quad (36)$$

where

$$V_1(w_i(t),t) = w_i^T(t)P_i(t)w_i(t)$$

$$V_2(w_i(t),t) = \sum_{l=1}^n \int_{-\tau_{kl}}^{0}\int_{t+\beta}^{t} \dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha d\beta$$

$$V_3(w_i(t),t) = \sum_{l=1}^n \int_{t-\tau_{kl}}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha.$$

The ith $(i = 2,3,\ldots,N)$ equation in the system (17) can be written as

$$\begin{aligned}\dot{\omega}_i(t) &= \left(D\bar{f}(s(t)) + c\lambda_i\Gamma\right)\omega_i(t)\\ &\quad - \frac{c\lambda_i}{n}\Gamma\sum_{l=1}^n \int_{t-\tau_{kl}}^{t} \dot{w}_i(\alpha)d\alpha\\ &\quad + \left(\Phi_i^T \otimes I_n\right) I(t)(\Phi \otimes I_n)\omega(t)\\ &\quad + \left(\Phi_i^T \otimes I_n\right) F(t), \qquad i = 2,3,\ldots,N \end{aligned} \quad (37)$$

and thus, the derivative of $V_1(w_i(t),t)$ satisfies

$$\begin{aligned}\dot{V}_1(w_i(t),t) &= w_i^T(t)\Big(\dot{P}_i(t) + P_i(t)\left(D\bar{f}(s(t)) + c\lambda_i\Gamma\right)\\ &\quad + \left(D\bar{f}(s(t)) + c\lambda_i\Gamma\right)^T P_i(t)\Big) w_i(t)\\ &\quad - \frac{2c\lambda_i}{n} w_i^T(t)P_i(t)\Gamma\sum_{l=1}^n \int_{t-\tau_{kl}}^{t} \dot{w}_i(\alpha)d\alpha\\ &\quad + 2w_i^T(t)P_i(t)\left(\Phi_i^T \otimes I_n\right) I(t)(\Phi_i \otimes I_n)w(t)\\ &\quad + 2w_i^T(t)P_i(t)\left(\Phi_i^T \otimes I_n\right) F(t). \end{aligned} \quad (38)$$

Defining $a(.)$, $b(.)$, and M in (10) as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$, and $M = (c\lambda_i/n)P_i(t)\Gamma$ for all $\alpha \in [t-\tau_{kl}, t]$ and then applying Lemma 2 result in

$$\begin{aligned}
&\dot{V}_1(w_i(t),t)\\
&\quad\leq w_i^T(t)\Big(\dot{P}_i(t)+P_i(t)\left(D\bar{f}(s(t))+c\lambda_i\Gamma\right)\\
&\qquad\qquad+\left(D\bar{f}(s(t))+c\lambda_i\Gamma\right)^T P_i(t)\Big)w_i(t)\\
&\qquad+nhw_i^T(t)X_iw_i(t)\\
&\qquad+2w_i^T(t)\left(Y_i-\frac{c\lambda_i}{n}P_i(t)\Gamma\right)\sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i(\alpha)d\alpha\\
&\qquad+\sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha\\
&\qquad+2w_i^T(t)P_i(t)\left(\Phi_i^T\otimes I_n\right)I(t)(\Phi_i\otimes I_n)w(t)\\
&\qquad+2w_i^T(t)P_i(t)\left(\Phi_i^T\otimes I_n\right)F(t).
\end{aligned}$$

$\dot{V}_1(w_i(t),t)$ can be further enlarged as

$$\begin{aligned}
&\dot{V}_1(w_i(t),t)\\
&\quad\leq w_i^T(t)\Big[\dot{P}_i(t)+P_i(t)D\bar{f}(s(t))+D\bar{f}(s(t))^T P_i(t)\\
&\qquad\qquad+nhX_i+2n\left(Y_i-\frac{c\lambda_i}{n}P_i(t)\Gamma\right)\Big]w_i(t)\\
&\qquad+2w_i^T(t)\left(\frac{c\lambda_i}{n}P_i(t)\Gamma-Y_i\right)\sum_{l=1}^{n}w_i(t-\tau_{kl})\\
&\qquad+\sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha\\
&\qquad+2w_i^T(t)P_i(t)\left(\Phi_i^T\otimes I_n\right)I(t)(\Phi_i\otimes I_n)w(t)\\
&\qquad+2w_i^T(t)P_i(t)\left(\Phi_i^T\otimes I_n\right)F(t).
\end{aligned}$$

Moreover, $\dot{V}_2(w_i(t),\ t) = \sum_{l=1}^{n}\int_{-\tau_{kl}}^{0}[D\bar{f}(s(t))\,\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)} + (\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)+(\Phi_i^T\otimes I_n)F(t)]^T Z_i[D\bar{f}(s(t))\omega_i(t)+c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}+(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)+(\Phi_i^T\otimes I_n)F(t)]d\alpha - \sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha$.

$\dot{V}_2(w_i(t),\ t) \leq nh\,[D\bar{f}(s(t))\,\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]^T Z_i\,[D\bar{f}(s(t))\,\omega_i(t) + c\lambda_i\,\Gamma\overline{\omega_i(t-\tau_j)}] + 2nh\,(D\bar{f}(s)\omega_i)^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega + 2nh(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T\otimes I_n)\,F(t) + 2nh\,(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i\,(\Phi_i^T\otimes I_n)\,I(t)\,(\Phi\otimes I_n)\,\omega + 2nh\,(c\lambda_i\,\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i\,(\Phi_i^T\otimes I_n)\,F(t) + 2nh\,((\Phi_i^T\otimes I_n)\,I(t)\,(\Phi\otimes I_n)\omega)^T Z_i\,(\Phi_i^T\otimes I_n)\,F(t) + nh\,((\Phi_i^T\otimes I_n)\,F(t))^T Z_i\,((\Phi_i^T\otimes I_n)\,F(t)) + nh\,((\Phi_i^T\otimes I_n)\,I(t)\,(\Phi\otimes I_n)\,\omega)^T Z_i\,((\Phi_i^T\otimes I_n)\,I(t)\,(\Phi\otimes I_n)\omega) - \sum_{l=1}^{n}\int_{t-\tau_{kl}}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha$.

$\dot{V}_3(w_i(t),t) \leq nw_i^T(t)Q_iw_i(t) - \sum_{l=1}^{n}w_i^T(t-\tau_k)Q_iw_i(t-\tau_{kl})$.

Then, we have $\sum_{k=1}^{3}\dot{V}_k(w_i(t),t) \leq w_i^T[\dot{P}_i(t)+P_i(t)D\bar{f}(s)+D\bar{f}(s)^T P_i(t)+nhX_i+2n(Y_i-(c\lambda_i/n)P_i(t)\Gamma)]w_i(t)+2w_i^T(t)((c\lambda_i/n)vP_i(t)\Gamma-Y_i)\sum_{l=1}^{n}w_i(t-\tau_{kl})+2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w(t)+2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)F(t)+2nh(D\bar{f}(s)\omega_i)^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega+nh[D\bar{f}(s(t))\omega_i(t)+c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]^T Z_i[D\bar{f}(s(t))\omega_i(t)+c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)}]+2nh(D\bar{f}(s(t))\omega_i(t))^T Z_i(\Phi_i^T\otimes I_n)F(t)+2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)+2nh(c\lambda_i\Gamma\overline{\omega_i(t-\tau_j)})^T Z_i(\Phi_i^T\otimes I_n)F(t)+2nh((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\,\omega(t))^T Z_i(\Phi_i^T\otimes I_n)\,F(t)+nh((\Phi_i^T\otimes I_n)\,I(t)(\Phi\otimes I_n)\,\omega(t))^T Z_i((\Phi_i^T\otimes I_n)\,I(t)\,(\Phi\otimes I_n)\,\omega(t))+nh((\Phi_i^T\otimes I_n)F(t))^T Z_i((\Phi_i^T\otimes I_n)F(t))+nw_i^TQ_iw_i-\sum_{l=1}^{n}w_i^T(t-\tau_{kl})Q_iw_i(t-\tau_{kl})$.

Applying the Young inequality, then, we have $2nh(c\lambda_i\Gamma\overline{w_i(t-\tau_j)})^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w \leq \overline{w_i(t-\tau_j)}^T\Pi_i^{-1}\overline{w_i(t-\tau_j)} + n^2h^2c^2\lambda_i^2w^T((\Phi\otimes I_n)^T I(t)(\Phi_i^T\otimes I_n)^T Z_i\Gamma\Pi_i\Gamma^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n))w(t)$ and $2nh(c\lambda_i\Gamma\overline{w_i(t-\tau_j)})^T Z_i(\Phi_i^T\otimes I_n)F(t) \leq \overline{w_i(t-\tau_j)}^T\Sigma_i^{-1}\overline{w_i(t-\tau_j)} + n^2h^2c^2\lambda_i^2F^T(t)(\Phi_i^T\otimes I_n)^T Z_i\Gamma\Theta_i\Gamma^T Z_i(\Phi_i^T\otimes I_n)F(t)$.

Applying these results to the inequality, then, we have $\dot{V} \leq \sum_{i=2}^{N}(w_i(t)/\overline{w_i(t-\tau_j)})^T\Xi(w_i(t)/\overline{w_i(t-\tau_j)}) + 2\mu(t)\beta + (\|w\|(2\gamma\beta+2nh\gamma\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{\max}(Z_i) + 2nh\mu(t)\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{\max}(Z_i) + nh\gamma^2\sum_{i=2}^{N}\lambda_{\max}(Z_i) + n^2h^2c^2\gamma^2\lambda_{\max}^{1/2}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{\max}(\Pi_i)\lambda_i^2\lambda_{\max}^2(Z_i) + (nhc\mu)^2\lambda_{\max}^{1/2}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{\max}\Theta_i\lambda_i^2\lambda_{\max}^2 Z_i)\|w\| + 2nh\gamma\sum_{i=2}^{N}\lambda_{\max}(Z_i)\mu(t)) + nh\mu^2(t)\sum_{i=2}^{N}\lambda_i^2\lambda_{\max}(Z_i)$.

Thus, when

$$\|w\| \geq \frac{2\mu(t)\beta + 2nh\gamma\sum_{i=2}^{N}\lambda_{\max}(Z_i)\mu(t)}{\varpi(t)}$$

we have

$$\dot{V} \leq -\delta\,\|w\|^2 + nh\mu^2(t)\sum_{i=2}^{N}\lambda_{\max}(Z_i)\lambda_i^2 \tag{39}$$

where Ξ is defined in (34), $\varpi(t) = -(2\gamma\beta + 2nh\gamma\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{\max}(Z_i) + 2nh\mu\|\Sigma_i\|\sum_{i=2}^{N}\lambda_{\max}(Z_i) + nh\gamma^2\sum_{i=2}^{N}\lambda_{\max}(Z_i) + n^2h^2c^2\gamma^2\lambda_{\max}^{1/2}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{\max}(\Pi_i)\lambda_i^2\lambda_{\max}^2(Z_i) + h^2c^2\mu^2(t)\lambda_{\max}^{1/2}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{\max}(\Theta_i)\lambda_i^2\lambda_{\max}^2(Z_i)) - \delta$. Then, according to Definition 1 and the Lyapunov stability theory, bounded consensus is ultimately achieved.

Remark 3.3: The aforementioned two bounded consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical nodes to the cases of nonidentical nodes. Because of the complexity of the consensus problems for nonidentical nodes, we only obtain here sufficient conditions instead of sufficient and necessary conditions. At the same time, the conditions obtained here are somewhat complicated and difficult to verify, but according to certain specific cases, we can construct an appropriate numerical-simulation example to verify the effectiveness of the proposed results. Comparing the aforementioned two theorems, it can be seen that the boundary of the convergence set and the maximum size of time delay can be evaluated, respectively.

Now, we will investigate the global bounded consensus problem for the following MAS which can be viewed as a special case of MAS (1):

$$\dot{x}_i(t) = f_i(x_i(t)) + u_i, \qquad i = 1, 2, \ldots, N \tag{40}$$

where all parameters have the same meanings as those in (1) and the unique difference is that in (40), every node has the same retardation time vector $(\tau_1, \tau_2, \ldots, \tau_n)$.

Repeating the similar process, we can transfer the consensus problem of MAS (40) to the stability problem of the $N-1$ of n-dimensional systems

$$\begin{aligned}\dot{\omega}_i(t) = {} & D\bar{f}(s(t)\omega_i(t) + c\lambda_i\Gamma\overline{\omega_i(t-\tau)} \\ & + \left(\Phi_i^T \otimes I_n\right) I(t)(\Phi \otimes I_n)\omega(t) \\ & + \left(\Phi_i^T \otimes I_n\right) F(t), \qquad i = 2, 3, \ldots, N. \end{aligned} \tag{41}$$

Similar to the analysis of Theorem 3.1, one can get the following corollary easily:

Corollary 3.1: Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$ and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$, and $b > 0$ such that

$$\begin{aligned} & a\|x(t)\|^2 \leq x^T(t)P_i(t)x(t) \leq b\|x(t)\|^2 \\ & \qquad \forall t \in R^+; \quad x \in R^n; \quad i = 2, 3, \ldots, N \end{aligned} \tag{42}$$

$$\begin{aligned} & \mu_2\left(\frac{1}{2}\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + \lambda_j\Gamma\right) + \zeta I \leq 0, \\ & \quad i = 1, 2, \ldots, N \end{aligned} \tag{43}$$

$$\|I(t)\| \leq \gamma, \qquad i = 1, 2, \ldots, N \tag{44}$$

and there exist an i and a j such that

$$\begin{pmatrix} -\lambda_j(\Gamma + \Gamma^T) & \lambda_i\Gamma \\ \lambda_i\Gamma^T & 0 \end{pmatrix} \leq 0 \tag{45}$$

for all $t \geq t_0$, where $i = 2, 3, \ldots, n$ and $j = 1, 2, \ldots, n$.

Let

$$\mu(t) = \|F(t)\| \tag{46}$$

be bounded and

$$\beta = \left(\sum_{i=2}^{N} \|P_i(t)\|^2\right)^{\frac{1}{2}}.$$

If $\zeta > 2\gamma\beta$, then the system (15) converges to the set

$$M = \left\{e(t) \middle| \|e(t)\| \leq \frac{2b}{a}\frac{\beta\overline{\lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\right\} \tag{47}$$

namely, $e(t) = x_i(t) - (1/N)\sum_{k=1}^{N} x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, and then, the MAS (1) achieves bounded consensus for any fixed time delay $\tau_k > 0 (k = 1, 2, \ldots, n)$.

IV. Example

In this section, a MAS consisting of 11 nonidentical agents will be constructed to demonstrate the efficiency of the results proposed in the previous section. For simplification, the delay-dependent result will be verified solely, and the delay-independent result can be verified similarly. The objective is to guarantee 11 agents to achieve bounded consensus, and the consensus curves are described as the average dynamics of 11 agents in a 3-D coordinate system.

The agent dynamics can be chosen as follows:

$$\dot{x}_i(t) = B_i x_i(t) + g(x_i(t)), \qquad i = 1, 2, \ldots, 11 \tag{48}$$

where

$$B_i = \begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix}, \quad i = 1, 2, \ldots, 6$$

$$B_i = \begin{pmatrix} -10-0.1\times(i-6) & 10+0.1\times(i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15+0.1\times(i-6) & 0 \end{pmatrix}, \quad i = 7, 8, \ldots, 11$$

$$g(x_i(t)) = \left(-9.5\sin\left(\frac{\pi x_{i1}(t)}{3.2} + \pi\right) 00\right)^T, \quad i = 1, 2, \ldots, 11.$$

The communication coupling matrix A and the inner coupling matrix are $A = (A_1^T A_2^T \ldots A_{11}^T)$, $A_1 = (-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1)$, $A_2 = (1\ -8\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1)$, $A_3 = (1\ 1\ -6\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1)$, $A_4 = (0\ 1\ 1\ -5\ 0\ 1\ 1\ 0\ 0\ 1\ 0)$, $A_5 = (1\ 1\ 0\ 0\ -6\ 0\ 1\ 1\ 1\ 1\ 0)$, $A_6 = (1\ 0\ 0\ 1\ 0\ -5\ 1\ 0\ 1\ 1\ 0)$, $A_7 = (1\ 1\ 0\ 1\ 1\ 1\ -7\ 1\ 0\ 1\ 0)$, $A_8 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ -5\ 0\ 1\ 1)$, $A_9 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ -6\ 1\ 1)$, $A_{10} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ -10\ 1)$, and $A_{11} = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ -6)$. $\Gamma = \mathrm{diag}\{2, 2, 2\}$, respectively.

Given the initial values of 11 agents as $(10\ 5\ -10)^T$, $(12\ 6\ -12)^T$, $(14\ 7\ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$, and $(-10\ 15\ 10)^T$, respectively, for any time delays $0 < \tau_{kl} \leq 0.061$, $1 \leq k$, and $l \leq 11$, the MAS will achieve consensus by means of verifying the conditions of Theorem 3.2 readily. The simulation results of the consensus errors are depicted in Figs. 1–3 for $c = 1$, and the communication time-delay matrix used to produce the present simulation curves are given as follows:

$$\begin{array}{c} \begin{pmatrix} 0.01 & 0.02 & 0.03 \\ 0.04 & 0.05 & 0.06 \\ 0.015 & 0.016 & 0.017 \\ 0.018 & 0.019 & 0.0195 \\ 0.021 & 0.022 & 0.023 \\ 0.024 & 0.025 & 0.026 \\ 0.027 & 0.028 & 0.029 \\ 0.031 & 0.032 & 0.033 \\ 0.034 & 0.035 & 0.036 \\ 0.037 & 0.038 & 0.039 \\ 0.045 & 0.055 & 0.061 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \text{1st agent} \\ \leftarrow \text{2nd agent} \\ \leftarrow \text{3rd agent} \\ \leftarrow \text{4th agent} \\ \leftarrow \text{5th agent} \\ \leftarrow \text{6th agent} \\ \leftarrow \text{7th agent} \\ \leftarrow \text{8th agent} \\ \leftarrow \text{9th agent} \\ \leftarrow \text{10th agent} \\ \leftarrow \text{11th agent.} \end{array}$$

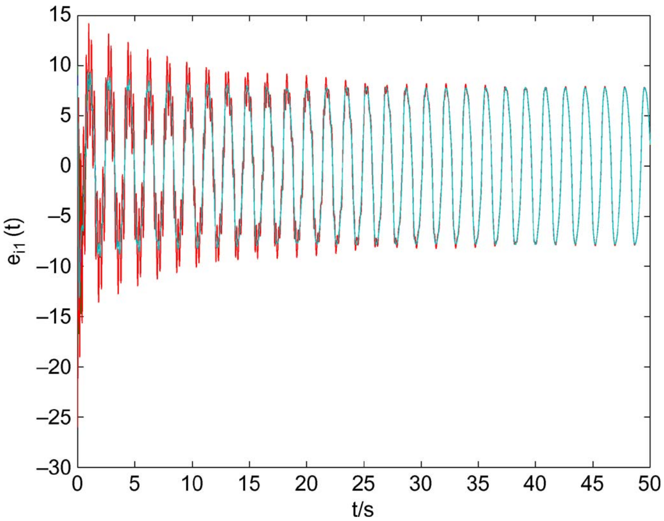

Fig. 1. Consensus errors $e_{i1}(t)$.

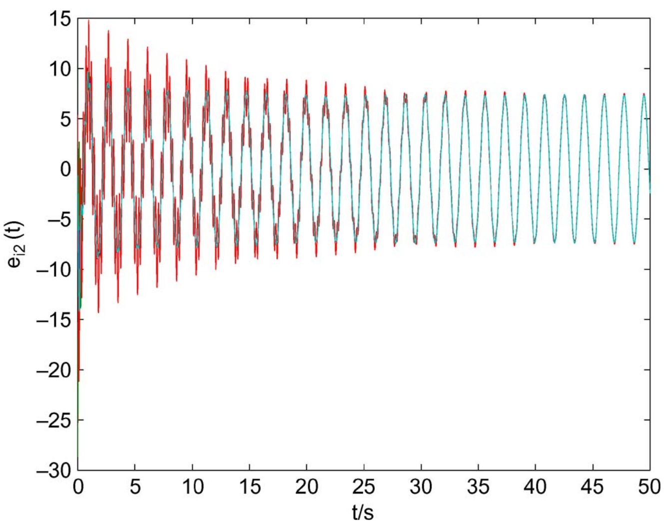

Fig. 2. Consensus errors $e_{i2}(t)$.

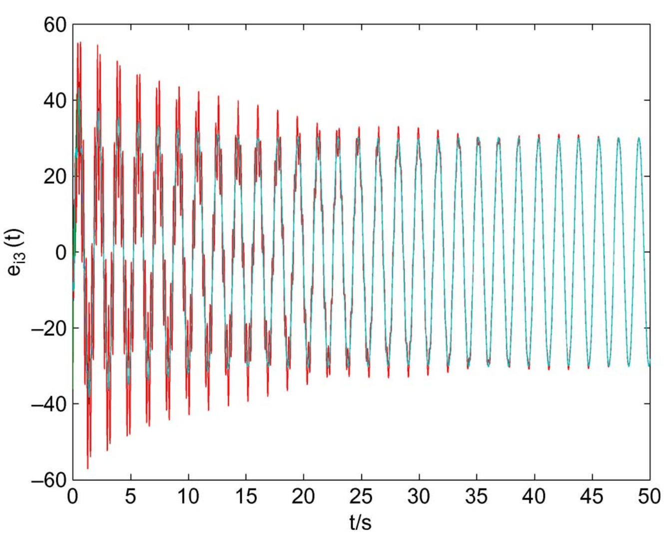

Fig. 3. Consensus errors $e_{i3}(t)$.

V. Conclusion

This paper has investigated the global consensus problems of MAS with different node dynamics. The derived criteria have been verified via theoretical analysis and numerical simulation. The consensus for the MAS has been achieved based on a series of transformations and the Lyapunov stability theorem. The methods presented here have several distinct features. First, they are very simple in form but are more effective in resolving the consensus problem with nonidentical agent dynamics. Second, the communication connection between agents is not direct, and there are constant time delays in the communication topology. It should be noted that the conditions are still restrictive. Further investigations will focus on relaxing these limitations.

Acknowledgment

The authors would like to thank the Associate Editor and all the anonymous reviewers for their valuable comments and constructive suggestions, which have led to the improvement of the presentation and the theoretical result of this paper.

References

[1] S. Zampieri, "Trends in networked control systems[C]," in *Proc. 17th World Congr. IFAC*, Seoul, Korea, Jul. 6–11, 2008, pp. 2886–2894.

[2] J. P. Desai, J. P. Ostrowski, and V. Kumar, "Modeling and control of formation of nonholonomic mobile robots," *IEEE Int. Trans. Robot. Autom.*, vol. 17, no. 6, pp. 905–908, Dec. 2001.

[3] W. Ren and R. W. Beard, "Formation feedback control for multiple spacecraft via virtual structures," *Inst. Elect. Eng. Proc. Control Theor. Appl.*, vol. 151, no. 3, pp. 357–368, May 2004.

[4] H. Yamaguchi, T. Arai, and G. Beni, "A distributed control scheme for multiple robotic vehicles to make group formations," *Robot. Auton. Syst.*, vol. 36, no. 4, pp. 125–147, Sep. 2001.

[5] Z. P. Wu, Z. H. Guan, and T. Li, "Group motion control of multi-agent systems based on complex network," in *Proc. 26th Chin. Control Conf.*, Hunan, China, Jul. 26–31, 2007, pp. 63–67.

[6] M. Porfiri, D. G. Roberson, and D. J. Stilwell, "Tracking and formation control of multiple autonomous agents: A two-level consensus approach," *Automatica*, vol. 43, no. 8, pp. 1318–1328, Aug. 2007.

[7] J. Cortés, "Global formation-shape stabilization of relative sensing networks," in *Proc. Amer. Control Syst. Conf.*, St. Louis, MO, Jun. 10–12, 2009, pp. 1460–1465.

[8] F. Xiao and L. Wang, "Consensus problems for high-dimensional multi-agent systems," *IET Control Theor. Appl.*, vol. 1, no. 3, pp. 830–837, May 2007.

[9] N. Cai, J. X. Xi, and Y. S. Zhong, "Swarm stability of high order LTI swarm systems," *IET Control Theor. Appl.*, vol. 5, no. 2, pp. 402–408, Jan. 2011.

[10] Y. Liu, Y. M. Jia, J. P. Du, and S. Y. Yuan, "Dynamic output feedback control for consensus of multi-agent systems: An H_∞ approach," in *Proc. Amer. Control Syst. Conf.*, St. Louis, MO, Jun. 10–12, 2009, pp. 4470–4475.

[11] W. W. Yu, G. R. Chen, M. Cao, and J. Kurths, "Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 3, pp. 881–891, Jun. 2010.

[12] Y. G. Hong, J. P. Hu, and L. X. Gao, "Tracking control for multi-agent consensus with an active leader and variable topology," *Automatica*, vol. 42, no. 7, pp. 1177–1182, 2006.

[13] H. Y. Yang, F. C. Wang, and S. Y. Zhang, "Consensus of second-order multi-agent systems with nonsymmetric interconnection and heterogeneous delays," *Int. J. Autom. Comput.*, vol. 8, no. 4, pp. 421–428, Nov. 2011.

[14] J. Yan, X. P. Guan, and F. X. Tan, "Target tracking and obstacle avoidance for multi-agent systems," *Int. J. Autom. Comput.*, vol. 7, no. 4, pp. 550–556, Nov. 2010.

[15] Z. Y. Meng, W. Ren, Y. C. Cao, and Z. You, "Leaderless and leader-following consensus with communication and input delays under a directed network topology," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 41, no. 1, pp. 75–88, Feb. 2011.

[16] F. Xiao and L. Wang, "Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays," *IEEE Trans. Autom. Control*, vol. 53, no. 8, pp. 1804–1816, Sep. 2008.

[17] J. R. Marden, G. Arslan, and J. S. Shamma, “Cooperative control and potential games,” *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 39, no. 6, pp. 1393–1407, Dec. 2009.

[18] T. Li and J. F. Zhang, “Decentralized tracking-type games for multi-agent systems with coupled ARX models: Asymptotic Nash equilibria,” *Automatica*, vol. 44, no. 3, pp. 713–725, 2008.

[19] V. Gazi, “Stability of an asynchronous swarm with time-dependent communication links,” *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 38, no. 1, pp. 267–274, Feb. 2008.

[20] E. S. Kazerooni and K. Khorasani, “Optimal consensus algorithms for cooperative team of agents subject to partial information,” *Automatica*, vol. 44, no. 11, pp. 2766–2777, Nov. 2008.

[21] E. Semsar-Kazerooni and K. Khorasani, “Optimal consensus seeking in a network of multiagent systems: An LMI approach,” *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 2, pp. 540–547, Apr. 2010.

[22] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks with switching topology and time-delays,” *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[23] M. Arcak, “Passivity as a design tool for group coordination,” *IEEE Trans. Autom. Control*, vol. 52, no. 8, pp. 1380–1390, Aug. 2007.

[24] F. Chen, Z. Q. Chen, L. Y. Xiang, Z. X. Liu, and Z. Z. Yuan, “Reaching a consensus via pinning control,” *Automatica*, vol. 45, no. 5, pp. 1215–1220, May 2009.

[25] Y. P. Tian and C. L. Liu, “Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations,” *Automatica*, vol. 45, no. 5, pp. 1347–1353, May 2009.

[26] P. Lin, Y. M. Jia, and L. Li, “Distributed robust H_∞ consensus control in directed networks of agents with time-delay,” *Syst. Control Lett.*, vol. 57, no. 8, pp. 643–653, Aug. 2008.

[27] P. A. Bliman and G. Ferrari-Trecate, “Average consensus problems in networks of agents with delayed communications,” *Automatica*, vol. 44, no. 8, pp. 1985–1995, Aug. 2008.

[28] J. Cortes, “Distributed algorithms for reaching consensus on general functions,” *Automatica*, vol. 44, no. 3, pp. 726–737, Mar. 2008.

[29] Z. K. Li, Z. S. Duan, G. R. Chen, and L. Huang, “Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint,” *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 57, no. 1, pp. 213–224, Jan. 2010.

[30] A. L. Barabási, R. Albert, H. Jeong, and G. Bianconi, “Power-law distribution of the world wide web,” *Science*, vol. 287, no. 5461, p. 2115a, Mar. 2000.

[31] S. H. Strogatz, “Exploring complex network,” *Nature*, vol. 410, no. 6825, pp. 268–276, Mar. 2001.

[32] V. N. Belykh, I. Belykh, and M. Hasler, “Connection graph stability method for synchronized coupled chaotic systems,” *Phys. D*, vol. 195, pp. 159–187, 2004.

[33] Z. Li and G. R. Chen, “Global synchronization and asymptotic stability of complex dynamical networks,” *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, no. 1, pp. 28–33, Jan. 2006.

[34] X. Li and G. R. Chen, “Synchronization and desynchronization of complex dynamical networks: An engineering viewpoint,” *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 11, pp. 1381–1390, Nov. 2003.

[35] X. F. Wang and G. R. Chen, “Complex networks: Small-world, scale-free and beyond,” *IEEE Circuits Syst. Mag.*, vol. 3, no. 1, pp. 6–20, 2003.

[36] J. H. Lü, X. H. Yu, G. R. Chen, and D. Z. Chen, “Characterizing the synchronizability of small-world dynamical networks,” *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 51, no. 4, pp. 787–796, Apr. 2004.

[37] J. Zhou, J. A. Lu, and J. H. Lü, “Adaptive synchronization of an uncertain complex dynamical network,” *IEEE Trans. Autom. Control*, vol. 51, no. 4, pp. 652–656, Apr. 2006.

[38] M. Y. Chen, “Some simple synchronization criteria for complex dynamical networks,” *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, no. 11, pp. 1185–1189, Nov. 2006.

[39] G. P. Jiang, W. K. S. Tang, and G. R. Chen, “A state-observer-based approach for synchronization in complex dynamical networks,” *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 53, no. 12, pp. 2739–2745, Dec. 2006.

[40] W. L. Lu, T. P. Chen, and G. R. Chen, “Synchronization analysis of linearly coupled systems described by differential equations with a coupling delay,” *Phys. D, Nonlinear Phenom.*, vol. 221, no. 2, pp. 118–134, Sep. 2006.

[41] J. Q. Lu, D. W. C. Ho, and J. Kurths, “Consensus over directed static networks with arbitrary finite communication delays,” *Phys. Rev. E*, vol. 80, no. 6, pp. 066121-1–066121-7, Dec. 2009.

[42] J. Q. Lu and D. W. C. Ho, “Globally exponential synchronization and synchronizability for general dynamical networks,” *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 2, pp. 350–361, Apr. 2010.

[43] M. Brede, “Synchrony-optimized networks of non-identical Kuramoto oscillators,” *Phys. Lett. A*, vol. 372, no. 15, pp. 2618–2622, 2008.

[44] J. Xiang and G. R. Chen, “On the V-stability of complex dynamical networks,” *Automatica*, vol. 43, no. 6, pp. 1049–1057, Jun. 2007.

[45] D. J. Hill and J. Zhao, “Global synchronization of complex dynamical networks with non-identical nodes,” in *Proc. 47th IEEE Conf. Decision Control*, Cancun, Mexico, Dec. 9–11, 2008, pp. 817–822.

[46] J. Zhao, D. J. Hill, and T. Liu, “Synchronization of dynamical networks with nonidentical nodes: Criteria and control,” *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 3, pp. 584–594, Mar. 2011.

[47] U. E. Vincent and J. A. Laoye, “Synchronization, anti-synchronization and current transports in non-identical chaotic ratchets,” *Phys. A, Stat. Mech. Appl.*, vol. 384, no. 2, pp. 230–240, Oct. 2007.

[48] R. K. Upadhyay and V. Rai, “Complex dynamics and synchronization in two non-identical chaotic ecological systems,” *Chaos Solitons Fractals*, vol. 40, no. 5, pp. 2233–2241, Jun. 2009.

[49] Q. Song, J. Cao, and W. Yu, “Second-order leader-following consensus of nonlinear multi-agent systems via pinning control,” *Syst. Control Lett.*, vol. 59, no. 9, pp. 553–562, Sep. 2010.

[50] Q. Song, J. Cao, and F. Liu, “Synchronization of complex dynamical networks with nonidentical nodes,” *Phys. Lett. A*, vol. 374, no. 4, pp. 544–551, Jan. 2010.

[51] W. He and J. Cao, “Consensus control for high order multi-agent systems,” *IET Control Theor. Appl.*, vol. 5, no. 1, pp. 231–238, Jan. 2011.

[52] C. C. Hua, X. P. Guan, and P. Shi, “Decentralized robust model reference adaptive control for interconnected time-delay systems,” *J. Comput. Appl. Math.*, vol. 193, no. 2, pp. 383–396, Sep. 2006.

[53] Y. M. Moon, P. Park, W. H. Kwon, and Y. S. Lee, “Delay-dependent robust stabilization of uncertain state-delayed systems,” *Int. J. Control*, vol. 74, pp. 1447–1455, 2001.

Wei-Song Zhong received the M.S. degree from Northeastern University, Shenyang, China, in 2005. He is currently working toward the Ph.D. degree in the Faculty of Advanced Technology, University of Glamorgan, Pontypridd, U.K.

His research interests include networked multi-agent systems, complex dynamical networks, and nonlinear control systems.

Guo-Ping Liu (M’97–SM’99–F’11) received the B.Eng. and M.Eng. degrees in automation from the Central South University of Technology, Changsha, China, in 1982 and 1985, respectively, and the Ph.D. degree in control engineering from University of Manchester Institute of Science and Technology, Manchester, Greater Manchester, U.K., in 1992.

Since 2001, he has been a Hundred-Talent-Program Visiting Professor with the Chinese Academy of Sciences, Beijing, China. Since 2004, he has been a Professor with the University of Glamorgan, Pontypridd, U.K., where he is the Chair of control engineering. Since 2008, he has been a Changjiang-Scholar Visiting Professor with Harbin Institute of Technology, Harbin, China. He has more than 400 publications on control systems and authored/coauthored seven books. He is the Editor-in-Chief of the *International Journal of Automation and Computing*. His main research areas include networked control systems, nonlinear-system identification and control, modeling and control of fuel-cell vehicles, advanced control of industrial systems, and multiobjective optimization and control.

Prof. Liu is a Fellow of Information Engineering Technology.

Clive Thomas received the B.Sc. (Honours) degree in electrical and electronic engineering from Swansea University, in 1975 and the Master of business administration degree from Cardiff University, in 1989.

He is a Principal Lecturer with the University of Glamorgan, Pontypridd, U.K., where he heads the Centre for Electronics Product Engineering. He joined the university following 20 years in industry working in signal processing and systems design roles. His current research interests include the industrial application of advanced control systems and the application of modeling methods to solve industrial application problems.

International Journal of Systems Science
2011, 1–12, iFirst

Global bounded consensus of multi-agent systems with non-identical nodes and communication time-delay topology

Wei-Song Zhong[a]*, Guo-Ping Liu[ab] and David Rees[a]

[a]*The Faculty of Advanced Technology, University of Glamorgan, CF37 1DL, Pontypridd, UK;* [b]*The CSIS Laboratory, Chinese Academy of Sciences, 100080 Beijing, China*

(*Received 4 May 2010; final version received 25 April 2011*)

This article investigates the global bounded consensus problem of networked multi-agent systems exhibiting nonlinear, non-identical node dynamics with communication time-delays. Globally bounded consensus conditions for both delay-independent and delay-dependent conditions based on the Lypunov–Krasovskii functional method are derived. The proposed consensus criteria ensures that all agents eventually move along the desired trajectories in the sense of boundedness. The proposed consensus criteria generalises the case of identical agent dynamics to the case of non-identical agent dynamics, and many related results in this area can be viewed as special cases of the above results. We finally demonstrate the effectiveness of the theoretical results by means of a numerical simulation.

Keywords: networked control systems; multi-agent systems; consensus; complex dynamical network

1. Introduction

Multi-agent systems (MAS) analysis involves the study of how the network architectures and interactions between network components influence global control goals. The research in this field can be categorised into two areas: one is to deal with the design of distributed estimation techniques which can be applied to the sensor networks, and the other is to deal with the control of mobile autonomous agents. In the latter case, each agent acts autonomously using information obtained over the network from other agents (Zampieri 2008). In both areas some important contributions have been made in recent years (Desai, Ostrowski, and Kumar 2001; Yamaguchi, Arai, and Beni 2001; Ren and Beard 2004; Porfiri, Roberson, and Stilwell 2007; Wu, Guan, and Li 2007; Cortés 2009).

The consensus problem requires an agreement to be reached that depends on the states of all agents. The topic has been studied across many fields of science and engineering (Hong, Hu, and Gao 2006; Kazerooni and Khorasani 2008; Li and Zhang 2008; Xiao and Wang 2008; Liu, Jia, Du, and Yuan 2009; Jiang, Yu, and Zhou 2011; Xiao, Chen, and Parhami 2011). Reza introduces two consensus criteria for networks with and without time-delays and provides convergence analysis for three kinds of MAS with fixed and switching topologies (Olfati-Saber and Murray 2004). A passivity-based design framework is developed to process the group coordination problem, where both fixed and time-varying communication structures are considered in Arack (2007). All agents reach a consensus if a small fraction of them are controlled by simple feedback control is proposed in Chen, Chen, Xiang, Liu, and Yuan (2009). The robust consensus problems of second-order MAS with diverse input delays are investigated and decentralised consensus conditions are obtained for the MAS with symmetric coupling weights based on frequency-domain analysis in Tian and Liu (2009). The consensus problem for directed MAS with external disturbances and model uncertainties for fixed and switching topologies are discussed in Lin, Jia, and Li (2008). The average consensus problem for undirected MAS having communication delays is studied and sufficient conditions are provided for the existence of average consensus under bounded communication delays in Bliman and Trecate (2008). A distributed algorithm that asymptotically achieved consensus is characterised and two discontinuous distributed algorithms that achieve max and min consensus are provided respectively in Cortes (2008). It is noted that the agent dynamics in most existing works are often restricted to linear and identical ones. Obviously, in practice, this is not always the case. The consensus problem of MAS with non-identical agent dynamics is much more complicated than the identical case and few results have been reported to date.

*Corresponding author. Email: wzhong@glam.ac.uk

ISSN 0020–7721 print/ISSN 1464–5319 online

DOI: 10.1080/00207721.2011.601346
http://www.informaworld.com

The similarity between the consensus of MAS and the synchronisation of complex networks shows us a way forward (Wu et al. 2007; Li, Duan, Chen, and Huang 2010). A complex network is a large set of interconnected dynamic nodes where specific representation is determined by the specific application. It has attracted tremendous attention in recent years (Barabási, Albert, Jeong, and Bianconi 2000; Strogatz 2001). Since the connection topology plays a key role in forming the behaviours of a complex network, researchers have examined a variety of connection topology and tried to better understand how the topology influences the network behaviour. Synchronisation is one of the key issues that affect network behaviour and has been extensively addressed and a vast number of papers on this topic has appeared based on complex networks with identical nodes (Li and Chen 2003, 2006; Wang and Chen 2003; Belykh, Belykh, and Hasler 2004; Lü, Yu, Chen, and Chen 2004; Chen 2006; Jiang, Tang, and Chen 2006; Zhou, Lu, and Lü 2006). As for the synchronisation of complex networks with non-identical nodes, some results have been proposed. A simulation based synchronisation study for non-identical Kuramoto oscillators was carried out in Brede (2008). A simple case where all non-identical nodes have the same equilibrium was considered in Xiang and Chen (2007) and a synchronisation criterion was given by constructing the same Lyapunov function for all the nodes. Hill and Zhao (2008) studied the synchronisation problem for a complex dynamical network with non-identical nodes and the proposed results extend the relevant asymptotic synchronisation criteria to this case. Several collective properties for coupled non-identical chaotic systems were discussed respectively in Vincent and Laoye (2007) and Upadhyay and Rai (2009). Therefore, if we use the ideas in the synchronisation problem of complex dynamical networks properly, then consensus problems are solvable.

Inspired by these early results, this article will focus on the global consensus problems of MAS, and the proposed consensus property is formulated in terms of certain boundedness of state errors. The behaviour of the MAS with non-identical agent dynamics is much more complicated than the identical case. Usually, neither common equilibrium for all agents exists even if each agent has an equilibrium, nor does a consensus manifold exist in the classical sense. Consensus of an MAS with identical agents is usually described in terms of (asymptotically) identical dynamical evolution of state variables of every agent in the MAS, which is easy to understand. However, this collective behaviour, called exact consensus no longer exists in the MAS with non-identical agents due to the difference between the dynamics of the agents. Furthermore, we can not decompose the MAS with non-identical agent dynamics into a number of lower dimensional systems exactly like the identical-agent case. Yet, an MAS with non-identical agents may still exhibit some kinds of consensus behaviours which are far from being fully understood, and very few results have been reported by now. Certain reasonable and satisfactory boundedness of state motions errors between different agents can be taken as useful consensus properties. Compared with many existing results, this article makes several significant advances. First, we generalise the related results for the case of identical agent dynamics to the case of non-identical agent dynamics and the proposed results cover the existing criteria of networks with identical agent dynamics as special cases. Second, we consider the communication time-delay among the agents and global consensus criteria are given based on solving a number of lower dimensional matrix inequalities and scalar inequalities, which generalise the criteria using the method of the master stability function for MAS with identical agents. Finally, globally bounded consensus conditions for both delay-independent and delay-dependent conditions based on the Lypunov–Krasovskii functional method are derived.

The rest of this article is organised as follows. A continuous-time MAS model with non-identical agent dynamics and communication time-delay is presented and some preliminaries are introduced in Section 2. The main results including delay-independent and delay-dependent bounded consensus criterion are derived in Section 3. In Section 4, a numerical simulation example is given to verify the effectiveness of the proposed results, followed by conclusions in Section 5.

2. Preliminaries

2.1. *Problem description*

Let $G=(\mathcal{V},\mathcal{A})$ be a graph of order N consisting of a set of vertices $\mathcal{V}=\{v_1,v_2,\dots,v_N\}$ and a set of edges $\mathcal{A}\subseteq\mathcal{V}\times\mathcal{V}$. An edge (v_j,v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbours of agent v_i is denoted by $\mathcal{N}_i=\{v_j\in\mathcal{V}:(v_j,v_i)\in\mathcal{A}\}$.

We consider an MAS consisting of N non-identical agents with communication delay:

$$\dot{x}_i(t)=f_i(x_i(t))+c\sum_{j\in\mathcal{N}_i}^{N}a_{ij}\Gamma x_j(t-\tau),\quad i=1,2,\dots,N, \tag{1}$$

where $x_i(t)=(x_{i1}(t),x_{i2}(t),\dots,x_{in}(t))^T\in R^n$ are the state variables of the agent v_i, $f_i(x_i(t)):R^n\to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix and $\gamma_{ij} \neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N\times N}$ (which is symmetric and irreducible) represents the communication topology relation of the MAS, and is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j\neq i} a_{ij}$. τ is a constant coupling delay which reflects the reality that the agent v_i can not obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field

$$\bar{f}(x(t)) = \frac{1}{N}\sum_{k=1}^{N} f_k(x(t)). \tag{2}$$

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t). \tag{3}$$

We now discuss the problem of global consensus for the MAS (1). The consensus problem formulation in this article is quite different from many others, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)$, $\forall i, j = 1, 2, \ldots, N$ as $t \to \infty$. The consensus problem here will be depicted instead via certain boundedness of $x_i(t) - x_j(t)$, $\forall i, j = 1, 2, \ldots, N$ as $t \to \infty$. This better reflects reality since it is impossible for MAS (1) to achieve exact consensus. To address this case, we will focus on making the states of all agents converge to a bounded set.

2.2. *Mathematical preliminaries*

Before stating the main results of this article, the following mathematical preliminaries are necessary.

Definition 1 (Hua, Guan, and Shi 2006)**:** The solution $x_i(t, t_0, \psi_i)$ of the MAS (1) is said to be uniformly ultimately bounded with respect to the bound ε if for each $\delta > 0$ there exists $T = T(\varepsilon, \delta) > 0$ independent of t_0 such that $\|x_i(t, t_0, \psi_i)\| \leq \varepsilon$ for all $t \geq t_0 + T$ when $\|x_i(t_0)\| < \delta$, where ψ_i is the initial value given as $x_i(t) = \psi_i$ for $t \in [t_0 - \tau, t_0]$, $i = 1, 2, \ldots, N$.

Lemma 1 (Hill and Zhao 2008)**:** *Assuming that the graph $G = (\mathcal{V}, \mathcal{A})$ is a strongly connected graph, then there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \ldots, \Phi_N)$, such that the adjacency matrix A satisfies*

$$\Phi^T A \Phi = \Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}, \tag{4}$$

where Φ_i is the i-th column of Φ with $\Phi_1 = (\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}})^T$ and $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$ are the eigenvalues of A.

Lemma 2 (Hill and Zhao 2008)**:** *Let $g(t)$ be a non-negative bounded function defined on R^+ and*

$$\Omega = \{x(t) \in R^n \mid \|x(t)\| \leq \overline{\lim}_{t\to\infty} g(t)\}. \tag{5}$$

Suppose there exist a strictly positive definite matrix $P(t) \in \mathcal{PC}^1_{n\times n}$ and a constant $\delta > 0$ such that the derivative of $V(x(t), t) = x^T(t)P(t)x(t)$ along the trajectory of the system

$$\dot{x}(t) = f(x(t), t), \quad x(t) \in R^n, \ t \in [0, \infty) \tag{6}$$

satisfies

$$\dot{V} \leq -\delta\|x(t)\|^2 \ \ \text{if} \ \ \|x(t)\| \geq g(t). \tag{7}$$

For any $t > 0$, let

$$Q_t = \{x(t) \mid V(x(t), t) \leq \sup_{y(s)\in\Omega, s\geq 0} \{V(y(s), s)\}\} \tag{8}$$

and

$$c = \overline{\lim}_{t\to\infty}(\max\{\|x(t)\| \mid x(t) \in Q_t\}). \tag{9}$$

Then, $x(t)$ converges to the set

$$M = \{x(t) \mid \|x(t)\| \leq c\}. \tag{10}$$

Lemma 3 (Moon, Park, Kwon, and Lee 2001)**:** *Assume that $a(\cdot) \in R^{n_a}$, $b(\cdot) \in R^{n_b}$ and $M(\cdot) \in R^{n_a\times n_b}$ are defined on an interval Ω. Then, for any matrices $X \in R^{n_a\times n_a}$, $Y \in R^{n_a\times n_b}$ and $Z \in R^{n_b\times n_b}$, the following inequality holds*:

$$-2\int_\Omega a^T(\alpha) M b(\alpha) \mathrm{d}\alpha \leq \int_\Omega \begin{pmatrix} a(\alpha) \\ b(\alpha) \end{pmatrix}^T \begin{pmatrix} X & Y - M \\ Y^T - M^T & Z \end{pmatrix} \begin{pmatrix} a(\alpha) \\ b(\alpha) \end{pmatrix} \mathrm{d}\alpha, \tag{11}$$

where

$$\begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} \geq 0.$$

We now introduce some notations and definitions.

Let $PC^1_{n\times n}$ be the linear space of the uniformly bounded continuous real matrix-valued functions defined on $[0, \infty)$. For any $P \in PC_{n\times n}$ the norm of P is defined by $\|P\| = \max_{0\leq t<\infty}\{\|P(t)\|\}$.

Let '$\otimes$' be Kronecker product.

3. Main results

Define the error vector

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \ldots, N. \tag{12}$$

Obviously, $\sum_{i=i}^{N} e_i = 0$ and $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i} a_{kj} \times \Gamma x_j(t-\tau) = 0$, then the MAS (1) can be rewritten in terms of e_i as

$$\begin{aligned}\dot{e}_i(t) &= f_i(s(t)+e_i(t)) - \frac{1}{N}\sum_{k=1}^{N} f_k(s(t)+e_k(t)) \\ &\quad + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma e_j(t-\tau).\end{aligned} \tag{13}$$

The following work will focus on simplifying the error MAS (13) by means of a series of transformations using a procedure similar to Hill and Zhao (2008).

Applying the Newton–Leibniz formula, error MAS (13) can be further written as

$$\begin{aligned}\dot{e}_i(t) &= D\bar{f}(s(t))e_i(t) + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma e_j(t-\tau) \\ &\quad + \int_0^1 (Df_i(s(t)+\tau e_i(t)) - D\bar{f}(s(t)))e_i(t)\mathrm{d}\tau \\ &\quad - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s(t)+\tau e_k(t))e_k(t)\mathrm{d}\tau \\ &\quad + f_i(s(t)) - \bar{f}(s(t)).\end{aligned} \tag{14}$$

If we consider the linearised MAS of (1), we have

$$\begin{aligned}\dot{e}_i(t) &= D\bar{f}(s(t))e_i(t) + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma e_j(t-\tau) + (Df_i(s(t)) \\ &\quad - D\bar{f}(s(t)))e_i(t) \\ &\quad - \frac{1}{N}\sum_{k=1}^{N} Df_k(s(t))e_k(t) + f_i(s(t)) - \bar{f}(s(t)).\end{aligned} \tag{15}$$

Let $e(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T$, then (14) becomes

$$\begin{aligned}\dot{e}(t) &= I_N \otimes D\bar{f}(s)e(t) + cA\otimes\Gamma e(t-\tau) + I(t)e(t) \\ &\quad - \frac{1}{N}H(t)e(t) + F(t),\end{aligned} \tag{16}$$

where

$$I(t) = \mathrm{diag}\left\{\int_0^1 (Df_1(s(t)+\tau e_1(t)) - D\bar{f}(s(t)))\mathrm{d}\tau \cdots \int_0^1 (Df_N(s(t)+\tau e_N(t)) - D\bar{f}(s(t)))\mathrm{d}\tau\right\},$$

$$H(t) = \begin{pmatrix} \int_0^1 Df_1(s(t)+\tau e_1(t))\mathrm{d}\tau & \cdots & \int_0^1 Df_N(s(t)+\tau e_N(t))\mathrm{d}\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s(t)+\tau e_1(t))\mathrm{d}\tau & \cdots & \int_0^1 Df_N(s(t)+\tau e_N(t))\mathrm{d}\tau \end{pmatrix},$$

$$F(t) = \begin{pmatrix} f_1(s(t)) - \bar{f}(s(t)) \\ \vdots \\ f_N(s(t)) - \bar{f}(s(t)) \end{pmatrix}.$$

Since A is symmetric and irreducible, according to Lemma 1, there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \ldots, \Phi_N)$, such that (4) is satisfied. This together with $\omega(t) = (\Phi^T \otimes I_n)e(t)$ gives

$$\begin{aligned}\dot{\omega}(t) &= (\Phi^T\otimes I_n)\dot{e}(t) \\ &= (\Phi^T\otimes I_n)(I_N\otimes D\bar{f}(s))(\Phi\otimes I_n)\omega(t) \\ &\quad + (\Phi^T\otimes I_n)(cA\otimes\Gamma)(\Phi\otimes I_n)\omega(t-\tau) \\ &\quad + (\Phi^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t) \\ &\quad - \frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)\omega(t) + (\Phi^T\otimes I_n)F(t).\end{aligned} \tag{17}$$

Note that

$$H(t) = \sqrt{N}\sum_{k=1}^{N}\left[(\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))\mathrm{d}\tau\right], \tag{18}$$

where $\bar{\Phi}_k$ stands for the matrix with its k-th column equals Φ_1 and the rest of its elements are zero, then we have

$$\begin{aligned}&\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n) \\ &\quad = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0}) \otimes \int_0^1 Df_k(s(t) \\ &\qquad + \tau e_k(t))\mathrm{d}\tau(\Phi\otimes I_n),\end{aligned} \tag{19}$$

where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the rest of its elements are zero.

Thus, a simple calculation gives

$$\begin{aligned}&\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n) \\ &\quad = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\begin{pmatrix}\Upsilon_k \\ 0\end{pmatrix} \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))\mathrm{d}\tau,\end{aligned} \tag{20}$$

where $\Upsilon_k \in R^{1\times N}$ and $0 \in R^{(N-1)\times N}$.

Therefore,

$$\begin{aligned}\dot{\omega}(t) &= I_N \otimes D\bar{f}(s)\omega(t) + c\Lambda \otimes \Gamma\omega(t-\tau)\\ &\quad + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) - \begin{pmatrix} * \\ 0 \end{pmatrix}\omega(t)\\ &\quad + (\Phi^T \otimes I_n)F(t). \end{aligned} \tag{21}$$

Since $\omega_1 \equiv 0$, we only need to consider $\omega_2(t), \omega_3(t), \ldots, \omega_N(t)$. Rewriting (21) in the component form, we have

$$\begin{aligned}\dot{\omega}_i(t) &= D\bar{f}(s(t)\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)\\ &\quad + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t)\\ &\quad + (\Phi_i^T \otimes I_n)F(t),\ i = 2, 3, \ldots, N. \end{aligned} \tag{22}$$

So far, we have transferred the consensus problem of MAS (1) to the stability problem of the $N-1$ of n-dimensional systems.

In the following, a time-independent global consensus criteria is derived for the MAS (1).

Theorem 3.1: *Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that*

$$\begin{aligned} a\|x(t)\|^2 &\leq x^T(t)P_i(t)x(t) + \int_{t-\tau}^{t} w_i^T(\alpha)Q_iw_i(\alpha)\mathrm{d}\alpha\\ &\leq b\|x(t)\|^2, \quad \forall t \in R^+,\ x \in R^n, i = 2, 3, \ldots, N, \end{aligned} \tag{23}$$

$$\begin{aligned} &\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^TP_i(t) + Q_i\\ &\quad + c^2\lambda_i^2P_i(t)\Gamma Q_i^{-1}\Gamma^TP_i(t) + \zeta I \leq 0, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{24}$$

$$\|I(t)\| \leq \gamma, \quad i = 1, 2, \ldots, N. \tag{25}$$

Then the system (16) *converges to the set*

$$M = \left\{ e(t) | \|e(t)\| \leq \frac{2b}{a}\frac{\beta\,\overline{\lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta} \right\} \tag{26}$$

for any fixed time delay $\tau > 0$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, $\beta = (\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}}$, $\zeta > 2\gamma\beta$ and $\mu(t) = \|F(t)\|$ is bounded. Furthermore, the MAS (1) *achieves bounded consensus for any fixed time delay $\tau > 0$.*

Proof: Choose the following Lyapunov–Krasovskii functional:

$$V(w_i(t), t) = \sum_{i=2}^{N} V_i(w_i(t), t), \tag{27}$$

$$\begin{aligned} V_i(w_i(t), t) &= w_i^T(t)P_i(t)w_i(t)\\ &\quad + \int_{t-\tau}^{t} w_i^T(\alpha)Q_iw_i(\alpha)\mathrm{d}\alpha, \quad i = 2, 3, \ldots, N. \end{aligned} \tag{28}$$

Differentiating (28) along the trajectory of (22) gives

$$\begin{aligned} \dot{V}_i(w_i(t), t) &= w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t))\\ &\quad + D\bar{f}(s(t))^TP_i(t) + Q_i)w_i(t)\\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t)\\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t)\\ &\quad + 2w_i^T(t)(c\lambda_iP_i(t)\Gamma)w_i(t-\tau)\\ &\quad - w_i^T(t-\tau)Q_iw_i(t-\tau). \end{aligned} \tag{29}$$

Applying the Young inequality to the equality (29) results in

$$\begin{aligned} \dot{V}_i(w_i(t), t) &\leq w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^TP_i(t)\\ &\quad + Q_i + c^2\lambda_i^2P_i(t)\Gamma Q_i^{-1}\Gamma^TP_i(t))w_i(t)\\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w(t)\\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t). \end{aligned} \tag{30}$$

Condition (24) implies that the first term on the right-hand side of (30) satisfies

$$\begin{aligned} &w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^TP_i(t) + Q_i\\ &\quad + c^2\lambda_i^2P_i(t)\Gamma Q_i^{-1}\Gamma^TP_i(t))w_i(t)\\ &\leq -\zeta\|w_i(t)\|^2. \end{aligned} \tag{31}$$

Applying condition (25), we know the second term on the right-hand side of (30) satisfies

$$\begin{aligned} &2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t)\\ &\quad \leq 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|. \end{aligned} \tag{32}$$

The third term on the right-hand side of (30) satisfies

$$2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i(t)\|. \tag{33}$$

Since $V(w(t), t) = \sum_{i=2}^{N} V_i(w_i(t), t)$, we have

$$\begin{aligned} \dot{V}(w(t), t) &= \sum_{i=2}^{N} \dot{V}_i(w_i(t), t)\\ &\leq \sum_{i=2}^{N}(-\zeta\|w_i(t)\|^2) + 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|\\ &\quad + 2\mu(t)\|P_i(t)\|\|w_i(t)\|\\ &= -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\|\\ &\quad + \mu(t))\sum_{i=2}^{N}\|w_i(t)\|\|P_i(t)\|\\ &\leq -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\|\\ &\quad + \mu(t))\|w(t)\|\left(\sum_{i=2}^{N}\|P_i(t)\|^2\right)^{\frac{1}{2}}\\ &= \|w(t)\|((2\gamma\beta - \zeta)\|w(t)\| + 2\beta\mu(t)). \end{aligned} \tag{34}$$

Thus, when

$$\|w(t)\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \tag{35}$$

we have

$$\dot{V}(w(t), t) \leq -\delta\|w(t)\|^2. \tag{36}$$

Applying Lemma 2 completes the proof.

Corollary 3.1: *We have an asymptotic consensus criterion in the classical sense when* $\overline{\lim}_{t\to\infty} \mu(t) = 0$. *In particular, we have* $\mu(t) \equiv 0$ *when all agents have the same dynamics, i.e.* $f_i(x_i(t)) = f(x(t))$. *In such a case, applying Theorem* 3.1 *to the linearised network* (15), *which is equivalent to taking* $\gamma = 0$ *in* (25), *immediately achieves the universal consensus criteria existing in many literatures. Therefore, Theorem* 3.1 *covers the existing criteria of networks with identical agent dynamics as a special case.*

Remark 1: The above result is a delay-independent globally consensus criterion and the ultimate convergence bound is evaluated by means of (26). Theorem 3.1 guarantees that all agents move along the desired trajectory described by $s(t)$ in terms of certain boundedness, i.e. the consensus achieved here is just approximate instead of exact, in fact, to achieve exact consensus is impossible for such a case.

Next, we will provide delay-dependent criterion for the proposed problem.

Theorem 3.2: *Suppose that* (23) *and* (25) *in Theorem* 3.1 *are satisfied. If there exist matrices* $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Pi_i > 0$, $\Pi_i > 0$, X_i, Y_i *and* Z_i *of appropriate dimensions such that*

$$\Xi = \begin{pmatrix} \dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + hX_i \\ \quad + Y_i^T + Y_i + Q_i + hD\bar{f}(s(t))^T Z_i D\bar{f}(s(t)) & c\lambda_i P_i(t)\Gamma - Y_i + hc\lambda_i D\bar{f}(s(t))^T Z_i \Gamma \\ c\lambda_i \Gamma^T P_i(t) - Y_i^T + hc\lambda_i \Gamma^T Z_i D\bar{f}(s(t)) & \Pi_i^{-1} + \Sigma_i^{-1} - Q_i + hc^2\lambda_i^2 \Gamma^T Z_i \Gamma \end{pmatrix} < 0, \tag{37}$$

where

$$\begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{38}$$

for $i = 2, 3, \ldots, N$, *then the MAS* (1) *will achieve bounded consensus for the time-invariant delay* $\tau \in [0, h]$ *for some* $h < \infty$.

Proof: Construct the following Lyapunov–Krasovskii functional:

$$V(w_i(t), t) = \sum_{i=2}^{N} \sum_{k=1}^{3} V_k(w_i(t), t), \tag{39}$$

where

$$V_1(w_i(t), t) = w_i^T(t) P_i(t) w_i(t),$$
$$V_2(w_i(t), t) = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha)\mathrm{d}\alpha\,\mathrm{d}\beta,$$
$$V_3(w_i(t), t) = \int_{t-\tau}^{t} w_i^T(\alpha) Q_i w_i(\alpha)\mathrm{d}\alpha.$$

The i-th $(i = 2, 3, \ldots, N)$ equation in system (22) can be written as

$$\begin{aligned}\dot{\omega}_i(t) &= (D\bar{f}(s(t)) + c\lambda_i\Gamma)\omega_i(t) - c\lambda_i\Gamma\int_{t-\tau}^{t}\dot{w}_i(\alpha)\mathrm{d}\alpha \\ &\quad + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) \\ &\quad + (\Phi_i^T \otimes I_n)F(t), \quad i = 2, 3, \ldots, N,\end{aligned} \tag{40}$$

and thus the derivative of $V_1(w_i(t), t)$ satisfies

$$\begin{aligned}\dot{V}_1(w_i(t), t) &= w_i^T(t)(\dot{P}_i(t) + P_i(t)(D\bar{f}(s(t)) + c\lambda_i\Gamma) \\ &\quad + (D\bar{f}(s(t)) + c\lambda_i\Gamma)^T P_i(t))w_i(t) \\ &\quad - 2c\lambda_i w_i^T(t)P_i(t)\Gamma\int_{t-\tau}^{t}\dot{w}_i(\alpha)\mathrm{d}\alpha \\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t).\end{aligned} \tag{41}$$

Defining $a(\cdot)$, $b(\cdot)$ and M in (11) as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$ and $M = c\lambda_i P_i(t)\Gamma$ for all $\alpha \in [t-\tau, t]$ and then applying Lemma 4 results in

$$\begin{aligned}\dot{V}_1(w_i(t), t) &\leq w_i^T(t)(\dot{P}_i(t) + P_i(t)(D\bar{f}(s(t)) + c\lambda_i\Gamma) \\ &\quad + (D\bar{f}(s(t)) + c\lambda_i\Gamma)^T P_i(t))w_i(t) \\ &\quad + \tau w_i^T(t)X_i w_i(t) + 2w_i^T(t)(Y_i - c\lambda_i P_i(t)\Gamma) \\ &\quad \times \int_{t-\tau}^{t}\dot{w}_i(\alpha)\mathrm{d}\alpha + \int_{t-\tau}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)\mathrm{d}\alpha \\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \\ &\leq w_i^T(t)[\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) \\ &\quad + hX_i + Y_i^T + Y_i]w_i(t) \\ &\quad + 2w_i^T(t)(c\lambda_i P_i(t)\Gamma - Y_i)w_i(t-\tau) \\ &\quad + \int_{t-\tau}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)\mathrm{d}\alpha \\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t).\end{aligned} \tag{42}$$

Moreover, since

$$\begin{aligned}\dot V_2(w_i(t),t) = \tau[D\bar f(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)\\ + (\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)\\ + (\Phi_i^T\otimes I_n)F(t)]^T Z_i[D\bar f(s(t))\omega_i(t)\\ + c\lambda_i\Gamma\omega_i(t-\tau)\\ + (\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)\\ + (\Phi_i^T\otimes I_n)F(t)] - \int_{t-\tau}^{t}\dot w_i^T(\alpha)Z_i\dot w_i(\alpha)\mathrm{d}\alpha,\end{aligned} \tag{43}$$

the above equality can be enlarged as

$$\begin{aligned}\dot V_2(w_i(t),t) \le h[D\bar f(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)]^T\\ \times Z_i[D\bar f(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)]\\ + 2h(D\bar f(s(t))\omega_i(t))^T Z_i(\Phi_i^T\otimes I_n)I(t)\\ \times(\Phi\otimes I_n)\omega(t) + 2h(D\bar f(s(t))\omega_i(t))^T\\ \times Z_i(\Phi_i^T\otimes I_n)F(t)\\ + 2h(c\lambda_i\Gamma\omega_i(t-\tau))^T Z_i(\Phi_i^T\otimes I_n)I(t)\\ \times(\Phi\otimes I_n)\omega(t) + 2h(c\lambda_i\Gamma\omega_i(t-\tau))^T\\ \times Z_i(\Phi_i^T\otimes I_n)F(t)\\ + 2h((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t))^T\\ \times Z_i(\Phi_i^T\otimes I_n)F(t) + h((\Phi_i^T\otimes I_n)F(t))^T\\ \times Z_i((\Phi_i^T\otimes I_n)F(t))\\ + h((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t))^T\\ \times Z_i((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t))\\ - \int_{t-\tau}^{t}\dot w_i^T(\alpha)Z_i\dot w_i(\alpha)\mathrm{d}\alpha.\end{aligned} \tag{44}$$

$$\dot V_3(w_i(t),t) = w_i^T(t)Q_iw_i(t) - w_i^T(t-\tau)Q_iw_i(t-\tau). \tag{45}$$

The derivative of $V(w_i(t), t)$ is

$$\begin{aligned}&\sum_{k=1}^{3}\dot V_k(w_i(t),t)\\ &\le w_i^T(t)[\dot P_i(t) + P_i(t)D\bar f(s(t)) + D\bar f(s(t))^T P_i(t)\\ &\quad + hX_i + Y_i^T + Y_i]w_i(t)\\ &\quad + 2w_i^T(t)(c\lambda_iP_i(t)\Gamma - Y_i)w_i(t-\tau)\\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w(t)\\ &\quad + 2w_i^T(t)P_i(t)(\Phi_i^T\otimes I_n)F(t)\\ &\quad + 2h(D\bar f(s(t))\omega_i(t))^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)\\ &\quad + h[D\bar f(s(t))\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau)]^T Z_i[D\bar f(s(t))\omega_i(t)\\ &\quad + c\lambda_i\Gamma\omega_i(t-\tau)]\\ &\quad + 2h(D\bar f(s(t))\omega_i(t))^T Z_i(\Phi_i^T\otimes I_n)F(t)\\ &\quad + 2h(c\lambda_i\Gamma\omega_i(t-\tau))^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)\\ &\quad + 2h(c\lambda_i\Gamma\omega_i(t-\tau))^T Z_i(\Phi_i^T\otimes I_n)F(t)\\ &\quad + 2h((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t))^T Z_i(\Phi_i^T\otimes I_n)F(t)\\ &\quad + h((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t))^T\\ &\quad \times Z_i((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t))\\ &\quad + h((\Phi_i^T\otimes I_n)F(t))^T Z((\Phi_i^T\otimes I_n)F(t))\\ &\quad + w_i^T(t)Q_iw_i(t) - w_i^T(t-\tau)Q_iw_i(t-\tau).\end{aligned} \tag{46}$$

Applying the Young inequality, we have

$$\begin{aligned}&2h(c\lambda_i\Gamma\omega_i(t-\tau))^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t)\\ &\le w_i^T(t-\tau)\Pi_i^{-1}w_i(t-\tau)\\ &\quad + h^2c^2\lambda_i^2w^T(t)((\Phi\otimes I_n)^T I(t)(\Phi_i^T\otimes I_n)^T\\ &\quad \times Z_i\Gamma\Pi_i\Gamma^T Z_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n))w(t)\end{aligned} \tag{47}$$

and

$$\begin{aligned}&2h(c\lambda_i\Gamma\omega_i(t-\tau))^T Z_i(\Phi_i^T\otimes I_n)F(t)\\ &\le w_i^T(t-\tau)\Sigma_i^{-1}w_i(t-\tau)\\ &\quad + h^2c^2\lambda_i^2F^T(t)(\Phi_i^T\otimes I_n)^T Z_i\Gamma\Sigma_i\Gamma^T Z_i(\Phi_i^T\otimes I_n)F(t).\end{aligned} \tag{48}$$

Applying (23) and (25) to the above inequality results in

$$\begin{aligned}\dot V(w_i(t),t) \le \sum_{i=2}^{N}\begin{pmatrix} w_i(t)\\ w_i(t-\tau)\end{pmatrix}^T \Xi \begin{pmatrix} w_i(t)\\ w_i(t-\tau)\end{pmatrix}\\ + \|w\|((2\gamma\beta + 2hr\|D\bar f(s(t))\|\\ \times\sum_{i=2}^{N}\lambda_{\max}(Z_i) + 2h\mu(t)\\ \times\|D\bar f(s(t))\|\sum_{i=2}^{N}\lambda_{\max}(Z_i)\\ + hr^2\sum_{i=2}^{N}\lambda_{\max}(Z_i) + h^2c^2r^2\lambda_{\max}^{\frac{1}{2}}(\Gamma\Gamma^T)\\ \times\sum_{i=2}^{N}\lambda_{\max}(\Pi_i)\lambda_i^2\lambda_{\max}^2(Z_i)\\ + h^2c^2\mu^2(t)\lambda_{\max}^{\frac{1}{2}}(\Gamma\Gamma^T)\\ \times\sum_{i=2}^{N}\lambda_{\max}(\Sigma_i)\lambda_i^2\lambda_{\max}^2(Z_i))\|w\|\\ + 2\mu(t)\beta + 2hr\sum_{i=2}^{N}\lambda_{\max}(Z_i)\mu(t))\\ + h\mu^2(t)\sum_{i=2}^{N}\lambda_i^2\lambda_{\max}(Z_i).\end{aligned} \tag{49}$$

Thus, when

$$\|w\| \geq \frac{2\mu(t)\beta + 2hr\sum_{i=2}^{N}\lambda_{\max}(Z_i)\mu(t)}{\left\{\begin{array}{l} -(2\gamma\beta + 2hr\|D\bar{f}(s(t))\|\sum_{i=2}^{N}\lambda_{\max}(Z_i) \\ +2h\mu(t)\|D\bar{f}(s(t))\|\sum_{i=2}^{N}\lambda_{\max}(Z_i) \\ +hr^2\sum_{i=2}^{N}\lambda_{\max}(Z_i) + h^2c^2r^2\lambda_{\max}^{\frac{1}{2}}(\Gamma\Gamma^T) \\ \times\sum_{i=2}^{N}\lambda_{\max}(\Pi_i)\lambda_i^2\lambda_{\max}^2(Z_i) \\ +h^2c^2\mu^2(t)\lambda_{\max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{\max}(\Sigma_i) \\ \times\lambda_i^2\lambda_{\max}^2(Z_i)) - \delta, \end{array}\right\}},$$

we have

$$\dot{V} \leq -\delta\|w\|^2 + h\mu^2(t)\sum_{i=2}^{N}\lambda_{\max}(Z_i)\lambda_i^2, \tag{50}$$

where Ξ is defined in (37). Thus, according to Definition 1 and Lyapunov stability theory, bounded consensus is ultimately achieved.

Remark 2: The above two bounded consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical nodes to the cases of non-identical nodes. Because of the complexity of the consensus problems for non-identical nodes, we only obtain here sufficient conditions instead of sufficient and necessary condition. At the same time, the conditions obtained here are somewhat complicated and difficult to verify, but according to certain specific cases, we can construct an appropriate numerical simulation example to verify the effectiveness of the proposed results. Comparing the above two theorems, it can be seen that the boundary of the convergence set and the maximum size of time delay can be evaluated respectively.

4. Example

In this section, we will construct an example to demonstrate the proposed results above. The problem is to guarantee 11 agents to follow desired curves in a 2-dimensional system of coordinate.

The agent dynamics can be chosen as follows:

$$\dot{x}_i(t) = B_i x_i(t) + g(x_i(t)), \quad i = 1, 2, \ldots, 11, \tag{51}$$

where

$$\left\{\begin{array}{l} B_i = \begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix}, \\ \quad i=1,2,\ldots,6, \\ B_i = \begin{pmatrix} -10-0.1\times(i-6) & 10+0.1\times(i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15+0.1\times(i-6) & 0 \end{pmatrix}, \\ \quad i=7,8,\ldots,11, \end{array}\right.$$

and

$$g(x_i(t)) = \left(-9.5\sin\left(\frac{\pi x_{i1}(t)}{3.2}+\pi\right)\ 0\ 0\right)^T, \quad i=1,2,\ldots,11.$$

The communication coupling matrix A and the inner coupling matrix are

$$A = \begin{pmatrix} -8 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -8 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -6 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & -7 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -5 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -10 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -6 \end{pmatrix}$$

and

$$\Gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Given the initial values of 11 agents as $(10\ 5\ {-10})^T$, $(12\ 6-12)^T$, $(14\ 7-14)^T$, $(16\ 8-16)^T$, $(18\ 9-18)^T$, $(20\ 10-20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$ and $(-10\ 15\ 10)^T$, respectively, we may verify the conditions of Theorem 3.2 readily. This demonstrates that the consensus of the MAS is achieved for any time delay $0 < \tau \leq 0.061$. Simulation results are depicted in Figures 1–5 for $\tau = 0.061$ and $c = 1$.

The simulation curves in Figure 1 show that the states of all agents are ultimately bounded stable. The average state trajectory $s(t)$ is chosen as the desired moving trajectory and is depicted in Figure 2. Figures 3–5 demonstrate that the state errors between

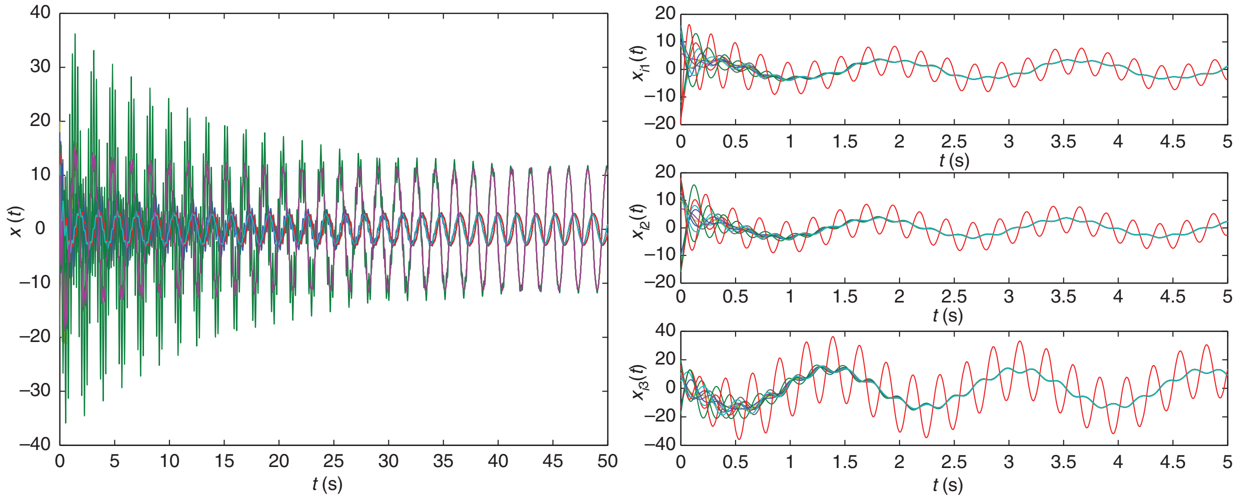

Figure 1. The dynamics of all agents with $t = 50$ and $t = 5$ s.

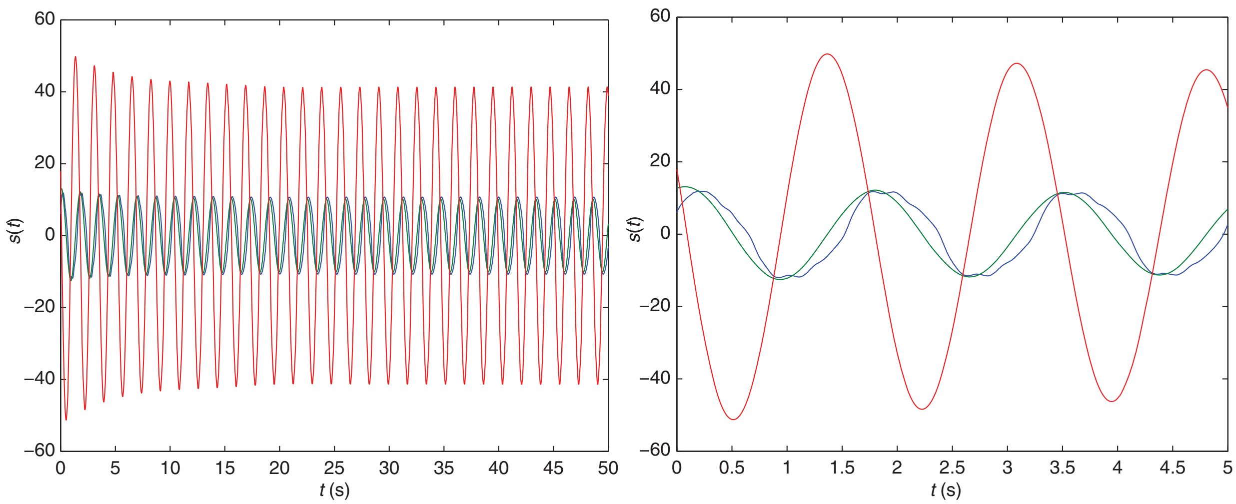

Figure 2. The average state trajectory $s(t)$ with $t = 50$ and $t = 5$ s.

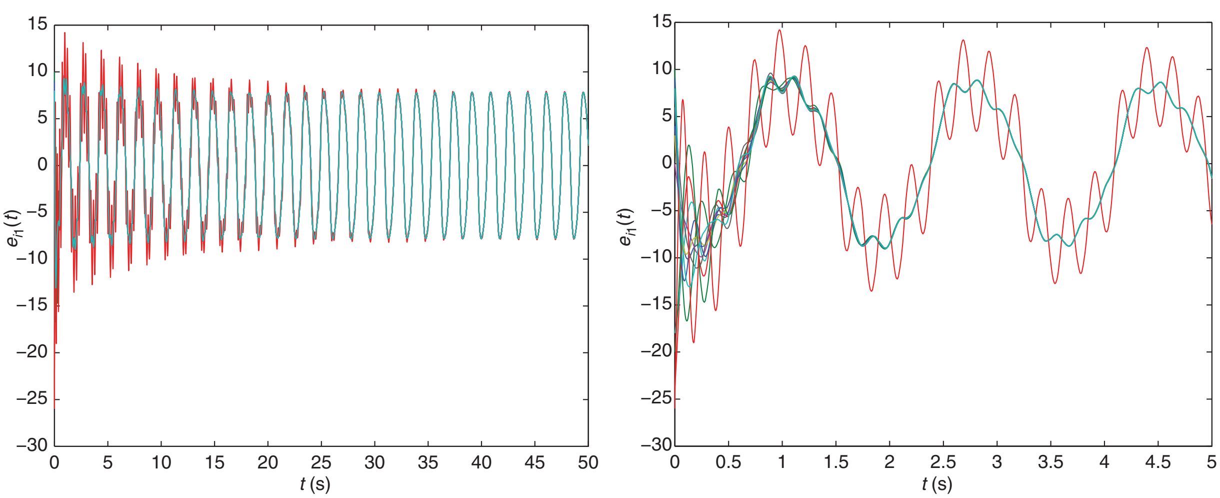

Figure 3. The consensus error dynamics for the first dynamic of each agent with $t = 50$ and $t = 5$ s.

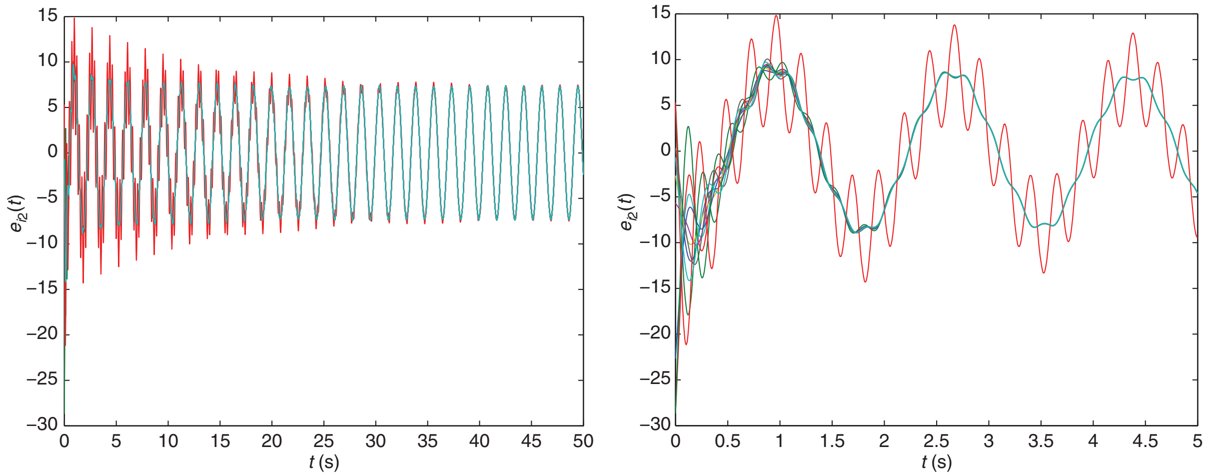

Figure 4. The consensus error dynamics for the second dynamic of each agent with $t = 50$ and $t = 5$ s.

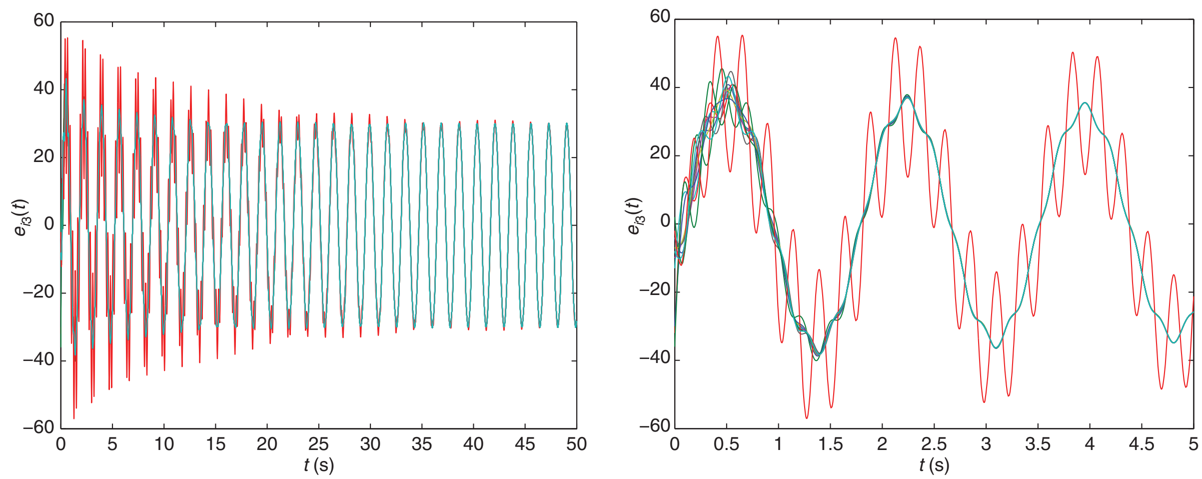

Figure 5. The consensus error dynamics for the third dynamic of each agent with $t = 50$ and $t = 5$ s.

each agent's states and the desired state trajectory, respectively, and the deviation systems are also ultimately bounded stable. These simulation curves show that all agents eventually move with the desired state trajectory in the sense of boundedness.

5. Conclusions

In this article, we have investigated the consensus problems of MAS with different node dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the MAS is achieved based on a series of transformations and Lyapunov stability theorem. The methods we present here have several distinct features. First, they are very simple in form, but are more effective to resolve the consensus problem with non-identical node dynamics. Second, the communication connection between agents are not direct, and there are constant time delays in the communication topology. It should be noted that the conditions are still restrictive and all the delays are the same. Further investigations will focus on relaxing these limitations.

Notes on contributors

Wei-Song Zhong received his BSc degree in Applied Mathematics from Shenyang University of Technology, Shenyang, China, in 2002, the MSc degree in Control Theory and Control Engineering from Northeastern University, China, in 2005. He is currently working towards PhD degree at the University of Glamorgan, Pontypridd, UK. His current research interests include the fields of networked multi-agent system and complex dynamical network.

Guo-Ping Liu received his BEng and MEng degrees in Electrical and Electronic Engineering from the Central South University of Technology, Changsha, China, in 1982 and 1985, respectively, and PhD degree in Control Engineering from UMIST, Manchester, UK, in 1992. He is a Chair of Control Engineering with the University of Glamorgan, Pontypridd, UK. He has been a Professor with the University of Glamorgan since 2004. His main research areas include networked control systems, modelling and control of fuel cell vehicles, advanced control of industrial systems, nonlinear system identification and control, and multiobjective optimisation and control. Dr Liu is the Editor-In-Chief of the *International Journal of Automation and Computing* and a Fellow of IEEE.

David Rees received his BSc (Honors) degree in Electrical Engineering from the University of Wales, College Swansea, UK, in 1967 and PhD degree from the Council of National Academic Awards, UK, in 1976. He was the Joint Editor of Industrial Digital Control Systems (Peter Peregrinus, 1988), and has contributed to numerous monographs. His current research interests include nonlinear modelling, system identification and networked control systems. Dr Rees was a recipient of an IEE Premium Award in 1996. He is a Fellow of the Institution of Engineering and Technology and a past Chairman of the IEE control Applications Professional Group.

References

Arack, M. (2007), 'Passivity as a Design Tool for Group Coordination', *IEEE Transactions on Automatic Control*, 52, 1380–1390.

Barabási, A.L., Albert, R., Jeong, H., and Bianconi, G. (2000), 'Power-law Distribution of the World Wide Web', *Science*, 287, 2115a.

Belykh, V.N., Belykh, I., and Hasler, M. (2004), 'Connection Graph Stability Method for Synchronised Coupled Chaotic Systems', *Physica D*, 195, 159–187.

Bliman, P.A., and Trecate, G.F. (2008), 'Average Consensus Problems in Networks of Agents with Delayed Communications', *Automatica*, 44, 1985–1995.

Brede, M. (2008), 'Synchrony-optimised Networks of Non-identical Kuramoto Oscillators', *Physics Letters A*, 372, 2618–2622.

Chen, M.Y. (2006), 'Some Simple Synchronzation Criteria for Complex Dynamical Networks', *IEEE Transactions on Circuits and Systems I*, 53, 1185–1189.

Chen, F., Chen, Z.Q., Xiang, L.Y., Liu, Z.X., and Yuan, Z.Z. (2009), 'Reaching a Consensus via Pinning Control', *Automatica*, 45, 1215–1220.

Cortes, J. (2008), 'Distributed Algorithms for Reaching Consensus on General Functions', *Automatica*, 44, 726–737.

Cortés, J. (2009), 'Global Formation-shape Stabilisation of Relative Sensing Networks', *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, 10–12 June, pp. 1460–1465.

Desai, J.P., Ostrowski, J.P., and Kumar, V. (2001), 'Modeling and Control of Formation of Nonholonomic Mobile Robust', *IEEE Transactions on Robotics and Automation*, 17, 905–908.

Hill, D.J., and Zhao, J. (2008), 'Global Synchronisation of Complex Dynamical Networks with Non-identical Nodes', in *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, 9–11 December, pp. 817–822.

Hong, Y.G., Hu, J.P., and Gao, L.X. (2006), 'Tracking Control for Multi-agent Consensus with an Active Leader and Variable Topology', *Automatica*, 42, 1177–1182.

Hua, C.C., Guan, X.P., and Shi, P. (2006), 'Decentralised Robust Model Reference Adaptive Control for Interconnected Time-delay Systems', *Journal of Computational Applied Mathematics*, 193, 383–396.

Jiang, G.P., Tang, W.K.S., and Chen, G.R. (2006), 'A State-observer-based Approach for Synchronisation in Complex Dynamical Networks', *IEEE Transactions on Circuits and Systems I*, 53, 2739–2745.

Jiang, H.B., Yu, J.J., and Zhou, C.G. (2011), 'Consensus of Multi-agent Linear Dynamic Systems via Impulsive Control Protocols', *International Journal of Systems Science*, 42, 967–976.

Kazerooni, E.S., and Khorasani, K. (2008), 'Optimal Consensus Algorithms for Cooperative Team of Agents Subject to Partial Information', *Automatica*, 44, 2766–2777.

Lü, J.H., Yu, X.H., Chen, G.R., and Chen, D.Z. (2004), 'Characterizing the Synchronisability of Small-world Dynamical Networks', *IEEE Transactions on Circuits and Systems I*, 51, 787–796.

Li, X., and Chen, G.R. (2003), 'Synchronisation and Desynchronisation of Complex Dynamical Networks: An Engineering Viewpoint', *IEEE Transactions on Circuits and Systems I*, 50, 1381–1390.

Li, Z., and Chen, G.R. (2006), 'Global Synchronisation and Asymptotic Stability of Complex Dynamical Networks', *IEEE Transactions on Circuits and Systems I*, 53, 28–33.

Li, Z.K., Duan, Z.S., Chen, G.R., and Huang, L. (2010), 'Consensus of Multiagent Systems and Synchronisation of Complex Networks: A Unified Viewpoint', *IEEE Transactions on Circuits and Systems-I*, 57, 213–224.

Li, T., and Zhang, J.F. (2008), 'Decentralised Tracking-type Games for Multi-agent Systems with Coupled ARX Models: Asymptotic Nash Equilibria', *Automatica*, 44, 713–725.

Lin, P., Jia, Y.M., and Li, L. (2008), 'Distributed Robust H_∞ Consensus Control in Directed Networks of Agents with Time-delay', *Systems and Control Letters*, 57, 643–653.

Liu, Y., Jia, Y.M., Du, J.P., and Yuan, S.Y. (2009), 'Dynamic Output Feedback Control for Consensus of Multi-agent Systems: An H_∞ Approach', in *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, 10–12 June, pp. 4470–4475.

Moon, Y.M., Park, P., Kwon, W.H., and Lee, Y.S. (2001), 'Delay-dependent Robust Stabilisation of Uncertain State-delayed Systems', *International Journal of Control*, 74, 447–1455.

Olfati-Saber, R., and Murray, R.M. (2004), 'Consensus Problems in Networks with Switching Topology and Time-delays', *IEEE Transactions on Automatic Control*, 49, 1520–1533.

Porfiri, M., Roberson, D.G., and Stilwell, D.J. (2007), 'Tracking and Formation Control of Multiple Autonomous Agents: A Two-level Consensus Approach', *Automatica*, 43, 1318–1328.

Ren, W., and Beard, R.W. (2004), 'Formation Feedback Control for Multiple Spacecraft via Virtual Structures', *IEE Proceedings of the Control Theory and Applications*, 151, 357–368.

Strogatz, S.H. (2001), 'Exploring Complex Network', *Nature*, 410, 268–276.

Tian, Y.P., and Liu, C.L. (2009), 'Robust Consensus of Multi-agent Systems with Diverse Input Delays and Asymmetric Interconnection Perturbations', *Automatica*, 45, 1347–1353.

Upadhyay, R.K., and Rai, V. (2009), 'Complex Dynamics and Synchronisation in Two Non-identical Chaotic Ecological Systems', *Chaos, Solutions and Fractals*, 40, 2233–2241.

Vincent, U.E., and Laoye, J.A. (2007), 'Synchronisation, Anti-synchronisation and Current Transports in Non-identical Chaotic Ratchets', *Physica A*, 384, 230–240.

Wang, X.F., and Chen, G.R. (2003), 'Complex Networks: Small-world, Scale-free and Beyond', *IEEE Circuits and Systems Magazine*, 3, 6–20.

Wu, Z.P., Guan, Z.H., and Li, T. (2007), 'Group Motion Control of Multi-agent Systems Based on Complex Network', in *Proceedings of the 26th Chinese Control Conference*, Hunan, China, 26–31 July, pp. 63–67.

Xiang, J., and Chen, G.R. (2007), 'On the V-stability of Complex Dynamical Networks', *Automatica*, 43, 1049–1057.

Xiao, W.J., Chen, W.D., and Parhami, B. (2011), 'On Necessary Conditions for Scale-freedom in Complex Networks, with Applications to Computer Communication Systems', *International Journal of Systems Science*, 42, 951–958.

Xiao, F., and Wang, L. (2008), 'Asynchronous Consensus in Continuous-time Multi-agent Systems with Switching Topology and Time-varying Delays', *IEEE Transactions on Automatic Control*, 53, 1804–1816.

Yamaguchi, H., Arai, T., and Beni, G. (2001), 'A Distributed Control Scheme for Multiple Robotic Vehicles to Make Group Formations', *Robotics and Autonomous Systems*, 36, 125–147.

Zampieri, S. (2008), 'Trends in Networked Control Systems', in *Proceedings of 17th World Congress of IFAC*, Seoul, Korea, 6–11 July, pp. 2886–2894.

Zhou, J., Lu, J.A, and Lü, J.H. (2006), 'Adaptive Synchronisation of an Uncertain Complex Dynamical Network', *IEEE Transactions on Automatic Control*, 51, 652–656.

:

Global Bounded Controlled Consensus of Networked Multi-Agents Systems with Non-Identical Dynamical Agents

Wei-Song Zhong * **Guo-Ping Liu** * **Clive Thomas** *

* *Faculty of Advanced Technology, University of Glamorgan, Pontypridd, CF37 1DL UK (e-mail: {wzhong,gpliu,ejcthoma}@glam.ac.uk).*

Abstract: This paper investigates the global bounded consensus problem of networked Multi-Agent Systems (MAS) exhibiting nonlinear, non-identical node dynamics with communication time-delays. Globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived. The proposed consensus criteria ensure that all agents eventually move along desired trajectories in terms of boundedness. The proposed controlled consensus criteria generalize the case of identical agent dynamics to the case of non-identical agent dynamics, and many related results of other researches in this area can be viewed as special cases of the above results. Finally, the effectiveness of the theoretical results is demonstrated by means of a numerical simulation.

1. INTRODUCTION

Networked Multi-Agent Systems (NMAS) investigations deal with the study of how network architecture and interactions between network components influence global control goals. This has attracted much attention due to the broad applications of NMAS in many areas. How to design appropriate protocols and algorithms such that the set of agents can realize common objective, such as consensus, is a critical problem, especially for the case of unreliable information exchange and communication delays, and some relevant important contributions have been made in recent years Zampieri [2008], Desai et al. [2001], Ren et al. [2004], Porfiri et al. [2007], Cortés [2009].

The consensus problem requires an agreement to be reached that depends on the state of all agents. The topic has been studied across many fields of science and engineering Liu et al. [2009], Hong et al. [2006], Xiao et al. [2008], Li et al. [2008], Kazerooni et al. [2008], Li et al. [2009], Olfati-Saber et al. [2004], Arack [2007], Chen et al. [2009], Tian et al [2009], Bliman et al. [2008], Cortés [2008]. It is noted that the agent dynamics in most existing works are often restricted to linear and identical ones. Obviously, in practice, this is not always the case. The controlled consensus problem of NMAS with nonlinear agent dynamics and communication delay are more complicated and just a few results have been made Hill et al. [2008]. In addition, most research in consensus problems usually assume that the final consensus value to be a constant, which may not be the case in the sense that the information state of each agent may be dynamically evolving in time according to some inherent dynamics. It is interesting to study controlled consensus problems where the final consensus value evolves with time or as a function of environmental dynamics.

The present paper will focus on the global consensus problems of NMAS based on pinning control methods, and the proposed controlled consensus property is formulated in terms of certain boundedness of state errors. Compared with existing related results, this paper make two significant advances. One is that we generalize the related results for the case of identical agent dynamics to the case of non-identical agent dynamics, and the other is we introduce pinning controllers to the selected agents.

The rest of this paper is organized as follows. A controlled continuous-time NMAS model with communication time-delay is presented in Section 2. The main results including pinning control and adaptive pinning control bounded consensus criterion are derived in Section 3 and 4 respectively. Section 5 gives a numerical simulation example to verify the effectiveness of the proposed results, followed by conclusions in Section 6.

2. PROBLEM DESCRIPTION

Let $G = (\mathcal{V}, \mathcal{A})$ be a graph of order N consisting of a set of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and a set of edges $\mathcal{A} \subseteq \mathcal{V}\times\mathcal{V}$. An edge (v_j, v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{A}\}$.

An NMAS consisting of N non-identical agents with communication delay is considered here:

$$\dot{x}_i = f_i(x_i) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma x_j(t-\tau), i = 1, 2, \cdots, N, \quad (1)$$

where $x_i = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix, and where $\gamma_{ij} \neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in$

$R^{N\times N}$ (which is symmetric and irreducible) represents the communication topology relation of the NMAS, and is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j\neq i} a_{ij}$. τ is a constant coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field $\bar{f}(x(t)) = \frac{1}{N}\sum_{k=1}^{N} f_k(x(t))$ with Jacobian $D\bar{f}_i(x(t))$.

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t). \tag{2}$$

Definition 1(Hua et al. [2006]): The solution $x_i(t, t_0, \psi_i)$ of the NMAS model (1) is said to be uniformly ultimately bounded with respect to the bound ε if for each $\delta > 0$ there exists $T = T(\varepsilon, \delta) > 0$ independent of t_0 such that $\|x_i(t, t_0, \psi_i)\| \leq \varepsilon$ for all $t \geq t_0 + T$ when $\|x_i(t_0)\| < \delta$, where ψ_i is the initial value.

Lemma 1 (Hill et al. [2008]): Let $g(t)$ be a non-negative bounded function defined on R^+ and

$$\Omega = \{x(t) \in R^n | \|x(t)\| \leq \overline{\lim_{t\to\infty}} g(t)\}. \tag{3}$$

Suppose there exists a strictly positive definite matrix $P(t) \in \mathcal{PC}^1_{n\times n}$ and a constant $\delta > 0$ such that the derivative of $V(x(t), t) = x^T(t)P(t)x(t)$ along the trajectory of the system

$$\dot{x}(t) = f(x(t), t), \quad x(t) \in R^n, t \in [0, \infty) \tag{4}$$

satisfies

$$\dot{V} \leq -\delta\|x(t)\|^2 \ \ \texttt{if} \ \ \|x(t)\| \geq g(t). \tag{5}$$

For any $t > 0$, let

$$Q_t = \{x(t) | V(x(t), t) \leq \sup_{y(s)\in\Omega, s\geq 0} \{V(y(s), s)\}\} \tag{6}$$

and

$$c = \overline{\lim_{t\to\infty}}(max\{\|x(t)\| | x(t) \in Q_t\}). \tag{7}$$

Then, $x(t)$ converges to the set

$$M = \{x(t) | \|x(t)\| \leq c\}. \tag{8}$$

In the rest of this paper, x, s, u, e, w, d_i and V denote $x(t)$, $s(t)$, $u(t)$, $e(t)$, $w(t)$, $d_i(t)$ and $V(w(t), t)$ respectively.

3. LINEAR FEEDBACK PINNING CONTROLLER

To achieve the goal, feedback control strategy will be applied on a small fraction δ $(0 < \delta \leq 1)$ of the agents in system (1). Suppose that nodes $i_1, i_2, \cdots, i_l$ are selected to be under control, where $l = [\delta N]$ stands for the smaller but nearest integer to the real number δN. This controlled NMAS can be described as

$$\begin{cases} \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j\in\mathcal{N}_i} a_{i_k j}\Gamma x_j(t-\tau) + u_{i_k}, 1 \leq k \leq l, \\ \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j\in\mathcal{N}_i} a_{i_k j}\Gamma x_j(t-\tau), l+1 \leq k \leq N. \end{cases} \tag{9}$$

The local linear negative feedback control law is chosen as follows:

$$u_{i_k} = -d_{i_k}(x_{i_k} - s), 1 \leq k \leq l \tag{10}$$

where the feedback gain $d_{i_k} > 0$.

Combine (9) and (10) and rearrange the order of the nodes in the network. Let the first l nodes be controlled, and $e_i = x_i - s$, $i = 1, 2, \cdots, N$. It's obvious that $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i} a_{kj}\Gamma x_j(t-\tau) = 0$ and $\sum_{i=1}^{N} e_i = 0$. Then by applying the Newton-Leibniz formula, error systems can be written as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau) \\ \quad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad + f_i(s) - \bar{f}(s) - d_i e_i, \qquad 1 \leq i \leq l, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau) \\ \quad + \int_0^1 (Df_i + \tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad + f_i(s) - \bar{f}(s), \qquad l+1 \leq i \leq N. \end{cases} \tag{11}$$

The following work will focus on simplifying the error systems (11) by means of a series of transformations using a procedure similar to Hill et al. [2008].

Define the following matrix

$$D = diag(D_1, D_2, \cdots, D_N) \in R^{nN\times nN},$$

where $D_i = diag\{-d_i, -d_i, \cdots, -d_i\} \in R^{n\times n}$.

Let $e = (e_1^T, e_2^T, \cdots, e_N^T)^T$, then (11) becomes

$$\dot{e} = \bar{\Sigma}(t)e + cA\otimes\Gamma e(t-\tau) + I(t)e - \frac{1}{N}H(t)e + F(t), \tag{12}$$

where $\bar{\Sigma}(t) = I_N \otimes D\bar{f}(s) + D$, $H^T(t) = (H_1^T(t), \cdots, H_N^T(t))$, $H_i(t) = (\int_0^1 Df_1(s+\tau e_1)d\tau, \cdots, \int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t) = (f_1^T(s) - \bar{f}^T(s), \cdots, f_N^T(s) - \bar{f}^T(s))$, $I(t) = diag\{\int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))d\tau\}, i = 1, 2, \cdots, N$.

Since A is symmetric and irreducible, according to Hill et al. [2008], there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1, \Phi_2, \cdots, \Phi_N)$. This together with $w(t) = (\Phi^T \otimes I_n)e$ gives

$$\begin{aligned}\dot{w} &= (\Phi^T \otimes I_n)\bar{\Sigma}(t)(\Phi \otimes I_n)w \\ &+ (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)w(t-\tau) \\ &+ (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ &- \frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)w + (\Phi^T \otimes I_n)F(t).\end{aligned} \tag{13}$$

Note that $H(t) = \sqrt{N}\sum_{k=1}^{N}(\mathbf{0} \ \cdots \ \mathbf{0} \ \bar{\Phi}_k \ \mathbf{0} \ \cdots \ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its k-th column equal to Φ_1 and the remaining elements are zero. Then we have $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}(\mathbf{0} \cdots \mathbf{0}\, I_k\, \mathbf{0} \cdots \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau(\Phi \otimes I_n)$, where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the remaining of its elements are zero.

Thus, a simple calculation gives $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\begin{pmatrix}\Upsilon_k \\ 0\end{pmatrix} \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k \in R^{1\times N}$ and $0 \in R^{(N-1)\times N}$. Therefore, $\dot{w} = \bar{\Sigma}(t)w + c\Lambda \otimes \Gamma w(t-\tau) + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w - \begin{pmatrix} * \\ 0\end{pmatrix} w + (\Phi^T \otimes I_n)F(t)$. Since $w_1 \equiv 0$, only $w_2, w_3, \cdots, w_N$ need to be considered. Rewriting in the component form we have

$$\begin{aligned}\dot{w}_i &= \Sigma_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau) + (\Phi_i^T \otimes I_n)F(t) \\ &+ (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w,\ i = 2,3,\cdots,N,\end{aligned} \tag{14}$$

where $\Sigma_i = \bar{D}f(s) + D_i$.

So far, the consensus problem of system (1) has been transferred to the stability problem of the $N-1$ of $n-$dimensional systems.

Theorem 1 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$\begin{aligned}a\|x\|^2 \leq x^T P_i(t)x + \int_{t-\tau}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \leq b\|x\|^2, \\ \forall t \in R^+,\ x \in R^n, i = 2,3,\cdots,N,\end{aligned} \tag{15}$$

$$\begin{aligned}&\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i \\ &+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \zeta I \leq 0,\ i = 1,2,\cdots,N,\end{aligned} \tag{16}$$

$$\|I(t)\| \leq \gamma,\ i = 1,2,\cdots,N. \tag{17}$$

Let

$$\mu(t) = \|F(t)\| \tag{18}$$

be bounded and

$$\beta = (\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}}, \tag{19}$$

if $\zeta > 2\gamma\beta$, then system (12) converges to the set

$$M = \{e | \|e\| \leq \frac{2b}{a}\frac{\beta\overline{lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\}, \tag{20}$$

for any fixed time delay $\tau > 0$, namely, $e(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^{N}x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, and then the NMAS (1) achieves bounded consensus for any fixed time delay $\tau > 0$.

Proof. Choose the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N} V_i, \tag{21}$$

$$V_i = w_i^T P_i(t)w_i + \int_{t-\tau}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha. \tag{22}$$

Differentiating (22) along the trajectory of (14) gives

$$\begin{aligned}\dot{V}_i &= w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i)w_i \\ &+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \\ &+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) + 2w_i^T(c\lambda_i P_i(t)\Gamma)w_i(t-\tau) \\ &- w_i^T(t-\tau)Q_i w_i(t-\tau).\end{aligned} \tag{23}$$

Applying the Young Inequality to the equality (23) results in

$$\begin{aligned}\dot{V}_i &\leq w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i \\ &+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) \\ &+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w.\end{aligned} \tag{24}$$

Condition (16) implies that the first term on the right hand side of (24) satisfies

$$\begin{aligned}&w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i \\ &+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i \leq -\zeta\|w_i\|^2.\end{aligned} \tag{25}$$

The second term on the right hand side of (24) satisfies

$$2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i\|. \tag{26}$$

Applying condition (17), the third term on the right hand side of (24) satisfies

$$2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \leq 2\gamma\|P_i(t)\|\|w_i\|\|w\|. \tag{27}$$

$V = \sum_{i=2}^{N} V_i$, then

$$\begin{aligned}\dot{V} &= \sum_{i=2}^{N}\dot{V}_i \\ &= -\zeta\|w\|^2 + 2(\gamma\|w\| + \mu(t))\sum_{i=2}^{N}\|w_i\|\|P_i(t)\| \\ &\leq -\zeta\|w\|^2 + 2(\gamma\|w\| + \mu(t))\|w\|(\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}} \\ &= \|w\|((2\gamma\beta - \zeta)\|w\| + 2\beta\mu(t)).\end{aligned} \tag{28}$$

Thus, when

$$\|w\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \tag{29}$$

gives

$$\dot{V} \leq -\delta\|w\|^2. \tag{30}$$

Applying Lemma 1 completes the proof.

4. ADAPTIVE PINNING CONTROLLER

In this section, globally consensus criteria will be derived via direct adaptive pinning control method. Without loss of generality, still assume that the first l agents are selected as pinned agents with the adaptive controllers:

$$\begin{cases} u_i = -d_i(x_i - s), & 1 \le i \le l, \\ \dot{d}_i = h_i e_i^T P_i(t) e_i, & \\ u_i = 0, & l+1 \le i \le N, \end{cases} \tag{31}$$

where constant $h_i > 0$ and positive definite matrix $P_i(t) \in R^{n\times n}$. Applying Newton-Leibniz formula, then the error NMAS can be rewritten as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau) \\ \quad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad + f_i(s) - \bar{f}(s) - d_i e_i, \qquad 1 \le i \le l, \\ \dot{d}_i = h_i e_i^T P_i(t) e_i, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau) \\ \quad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad + f_i(s) - \bar{f}(s), \qquad l+1 \le i \le N. \end{cases} \tag{32}$$

Repeating a similar procedure to the previous subsection, the controlled consensus problem of system (1) is equivalent to the stability problem of the following $N-1$ of n-dimensional systems.

$$\begin{cases} \dot{w}_i = D\bar{f}(s(t))w_i - d_i w_i + c\lambda_i\Gamma w_i(t-\tau) \\ \qquad + (\Phi_i^T \otimes I_n)I(t)(\Phi\otimes I_n)w \\ \qquad + (\Phi_i^T \otimes I_n)F(t), \qquad 2 \le i \le l, \\ \dot{d}_i = h_i w_i^T P_i(t) w_i, \\ \dot{w}_i = D\bar{f}(s)w_i + c\lambda_i\Gamma w_i(t-\tau) \\ \qquad + (\Phi_i^T \otimes I_n)I(t)(\Phi\otimes I_n)w \\ \qquad + (\Phi_i^T \otimes I_n)F(t), \qquad l+1 \le i \le N, \end{cases} \tag{33}$$

where w_i, w, Φ, Φ_i, $I(t)$ and $F(t)$ are the same as the previous subsection.

Theorem 2 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\bar{\zeta} > 0$, $\gamma \ge 0$, $a > 0$ and $b > 0$ such that

$$a\|x\|^2 \le x_i^T P_i(t)x_i + \int_{t-\tau}^{t} x_i^T(\alpha)Q_i x_i(\alpha)d\alpha + \frac{(d_i-d)^2}{h_i} \le b\|x\|^2, \forall t\in R^+,\ x\in R^n, i = 2,3,\cdots,N, \tag{34}$$

$$\dot{P}_i(t) + P_i(t)D\bar{f}(s) + (D\bar{f}(s))^T P_i(t) + Q_i - 2dP_i(t) + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \bar{\zeta} I \le 0,\ i = 1,2,\cdots,N, \tag{35}$$

(17) and $\bar{\zeta} > 2\gamma\beta$ are satisfied, then the system (12) converges to the set (20) for any fixed time delay $\tau > 0$, where $\mu(t)$ and β are the same as in (18) and (19) respectively, $\bar{\delta} > 0$ is any constant satisfying $\bar{\delta} < \bar{\zeta} - 2\gamma\beta$, and then the NMAS (1) achieves bounded consensus for any fixed time delay $\tau > 0$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N} V_i + \sum_{i=2}^{l}\frac{(d_i-d)^2}{h_i}, \tag{36}$$

where

$$\begin{cases} V_i = w_i^T P_i(t) w_i + \int_{t-\tau}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \\ \qquad + \frac{(d_i-d)^2}{h_i}, \qquad 2 \le i \le l, \\ V_i = w_i^T P_i(t) w_i + \int_{t-\tau}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha, l+1 \le i \le N, \end{cases} \tag{37}$$

where d is a positive constant to be determined.

Differentiating (37) along the trajectory of (33) gives

$$\begin{aligned} \dot{V}_i &= w_i^T(\dot{P}_i(t) + P_i(t)D\bar{f}(s) + (D\bar{f}(s))^T P_i(t) + Q_i \\ &\quad - 2dP_i(t))w_i + 2w_i^T P_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w \\ &\quad + 2w_i^T P_i(t)(\Phi_i^T\otimes I_n)F(t) + 2w_i^T(c\lambda_i P_i(t)\Gamma)w_i(t-\tau) \\ &\quad - w_i^T(t-\tau)Q_i w_i(t-\tau). \end{aligned} \tag{38}$$

The remaining part of the proof is similar to that of Theorem 1, so is therefore omitted here. This completes the proof.

5. EXAMPLES

To demonstrate the theoretical results obtained above, an NMAS consisting of 12 agents is constructed and is described as follows

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma x_j(t-\tau), \tag{39}$$

where $f_i(x_i(t)) = B_i x_i(t) + g(x_i(t))$, $B_i(i = 1,2,\cdots,6)$ and $B_i(i = 7,8,\cdots,12)$ are chosen as follows:

$$\begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix},$$

$$\begin{pmatrix} -10-0.1\times(i-6) & 10+0.1\times(i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15+0.1\times(i-6) & 0 \end{pmatrix},$$

and

$$g(x_i(t)) = (-9.5sin(\frac{\pi x_{i1}(t)}{3.2} + \pi)\ 0\ 0)^T, \quad i = 1,2,\cdots,12.$$

The communication coupling matrix $C = (C_1^T C_2^T \cdots C_{12}^T)$, $C_1 = (-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0)$, $C_2 = (1\ -8\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0)$, $C_3 = (1\ 1\ -7\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1)$, $C_4 = (0\ 1\ 1\ -6\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1)$, $C_5 = (1\ 1\ 0\ 0\ -6\ 0\ 1\ 1\ 1\ 1\ 0\ 0)$, $C_6 = (1\ 0\ 0\ 1\ 0\ -5\ 1\ 0\ 1\ 1\ 0)$, $C_7 = (1\ 1\ 0\ 1\ 1\ 1\ -7\ 1\ 0\ 1\ 0)$, $C_8 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ -6\ 0\ 1\ 1\ 1)$, $C_9 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ -7\ 1\ 1\ 1)$, $C_{10} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ -10\ 1)$, $C_{11} = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ -7\ 1)$,

$C_{12} = (0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ \ -5)$. $\Gamma = diag\{2, 2, 2\}$, respectively, where the matrix A is produced by means of the Scale-Free network program.

Design the following controllers

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k}(t) - s(t)), & i_k = 1, 2 \texttt{ and } 10, \\ u_{i_k} = 0, & \texttt{else}, \end{cases}$$

with $d_1 = 0.5$, $d_2 = 0.5$, $d_{10} = 0.5$ and

$$\begin{cases} u_{i_k} = -d_{i_k}(t)(x_{i_k}(t) - s(t)), & i_k = 1, 2 \texttt{ and } 10, \\ \dot{d}_{i_k}(t) = h_{i_k} e_{i_k}^T P_{i_k}(t) e_{i_k}, & \\ u_{i_k} = 0, & \texttt{else}, \end{cases}$$

with $h_1 = 0.1$, $h_2 = 0.2$, $h_{10} = 0.3$, $s(t)$ can then be evaluated by simulation.

Given the initial values of 12 agents as $(10\ 5\ \ -10)^T$, $(12\ 6\ \ -12)^T$, $(14\ 7\ \ -14)^T$, $(16\ 8\ \ -16)^T$, $(18\ 9\ \ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$, $(-10\ 15\ 10)^T$, $(-8\ 16\ 8)^T$ respectively and $P_{i_k}(t) = I_3$, $d_1(0) = 1$, $d_2(0) = 1$, $d_{10}(0) = 1$. We may verify the conditions of Theorem 1 and Theorem 2 readily. This demonstrates the bounded consensus of the NMAS is achieved for any time delay $0 < \tau \leq 0.06$. Simulation results are depicted in Fig.1 to Fig.8 for $\tau = 0.06$ and $c = 1$.

6. CONCLUSION

In this paper, the controlled consensus problems of NMAS with different agent dynamics have been investigated. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on pinning control and adaptive pinning control methods. It should be noted that the conditions are still restrictive and all the delays are the same. Further investigations will focus on relaxing these limitations.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China (No.61028010).

REFERENCES

S. Zampieri. Trends in networked control systems. *Proceedings of 17th World Congress of IFAC*, July 6-11, Seoul, Korea, pages 2886–2894, 2008.

J.P. Desai, J.P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust. *IEEE Int. Trans. on Robotics and Automation*, 17(6): 905–908, 2001.

W. Ren and R.W. Beard. Formation feedback control for multiple spacecraft via virtual structures. *IEE Proc. Control Theory and Applications*, 151(3): 357–368, 2004.

H. Yamaguchi, T. Arai and G. Beni. A. distributed control scheme for multiple robotic vehicles to make group formations. *Robitics and Autonomous Systems*, 36: 125–147, 2001.

Z.P. Wu, Z.H. Guan, T. Li. Group motion control of multi-agent systems based on complex network. *Proceedings of the 26th Chinese Control Conference*, July 26-31, Hunan, China, pages 63–67, 2007.

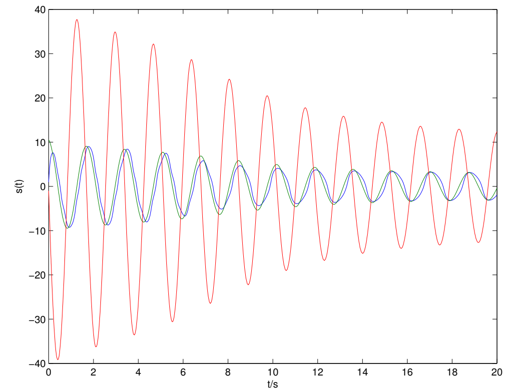

Fig.1. Desired agent dynamics under pinning control.

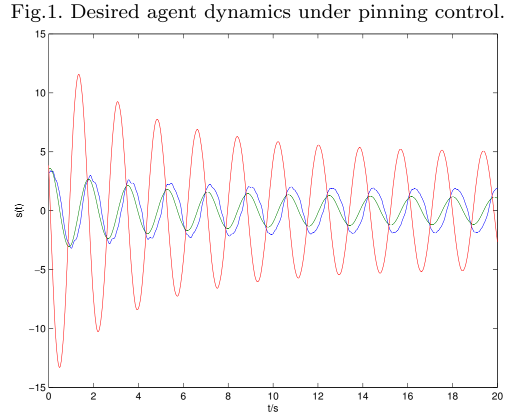

Fig.2. Desired agent dynamics under adaptive pinning control.

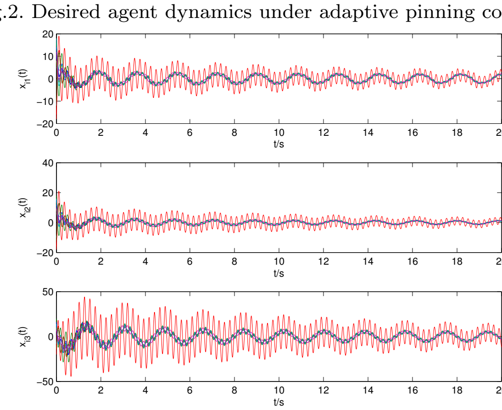

Fig.3. All agent dynamics under pinning control.

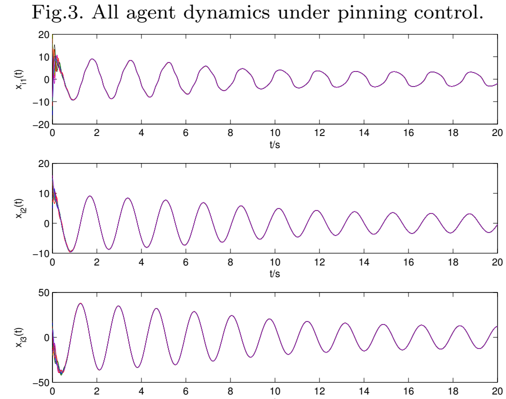

Fig.4. All agent dynamics under adaptive pinning control.

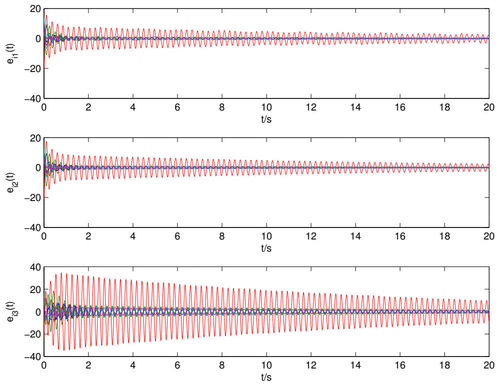

Fig.5. All agent error dynamics under pinning control.

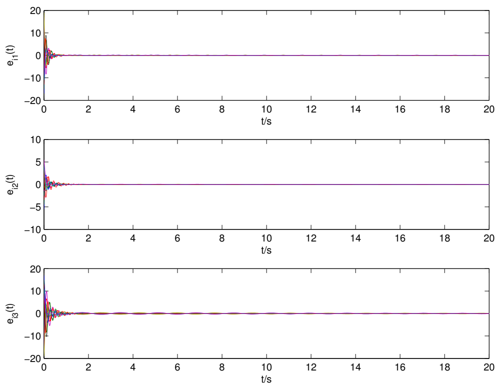

Fig.6. All agent error dynamics under adaptive pinning control.

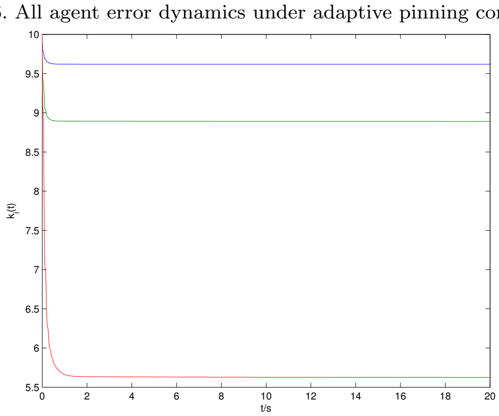

Fig.7. Adaptive gain curves.

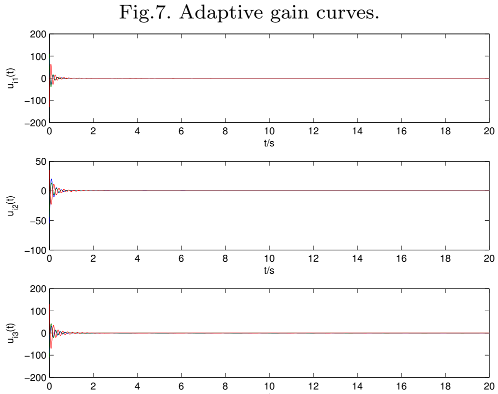

Fig.8. Adaptive pinning controllers curves.

M. Porfiri, D.G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach. *Automatica*, 43: 1318-1328, 2007.

J. Cortés. Global formation-shape stabilization of relative sensing networks. *2009 American Control Systems*, June 10-12, Hyatt Regency Riverfront, St. Louis, USA, pages 1460–1465, 2009.

Y. Liu, Y.M. Jia, J.P. Du and S.Y. Yuan. Dynamic output feedback control for consensus of multi-agent systems: an H_∞ approach. *2009 American Control Systems*, June 10-12, Hyatt Regency Riverfront, St. Louis, USA, pages 4470–4475, 2009.

Y.G. Hong, J.P. Hu and L.X. Gao. Tracking control for multi-agent consensus with an active leader and variable topology. *Automatica*, 42: 1177–1182, 2006.

F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Transactions on Automatic Control*, 53(8): 1804–1816, 2008.

T. Li and J.F. Zhang. Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash equilibria. *Automatica*, 44: 713–725, 2008.

E.S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information. *Automatica*, 44: 2766–2777, 2008.

R. Olfati-Saber and R.M. Murray. Consensus problmes in networks with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9): 1520–1533, 2004.

M. Arack. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8): 1380–1390, 2007.

F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu and Z.Z. Yuan. Reaching a consensus via pinning control. *Automatica*, 45: 1215–1220, 2009.

Y.P. Tian and C.L Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations. *Automatica*, 45: 1347–1353, 2009.

P.A. Bliman and G.F. Trecate. Average consensus problems in networks of agents with delayed communications. *Automatica*, 44: 1985–1995, 2008.

J. Cortés. Distributed algorithms for reaching consensus on general functions. *Automatica*, 44(3): 726–737, 2008.

Z.K. Li, Z.S. Duan, G.R. Chen and L. Huang. Consensus of multi-agent systems and synchronization of complex networks: a unified viewpoint. *IEEE Transactions on Circuits and Systems-I*, 57(1): 213–224, 2009.

Z.G. Hou, L. Cheng and M. Tan. Decentralized robust adaptive control for the multiagent system consensus problem using neural networks. *IEEE Transactions on Systems, Man, and Cybernetics-Part B: Cybernetics*, 39(3): 636–647, 2009.

C.C. Hua, X.P. Guan and P. Shi. Decentralized robust model reference adaptive cotnrol for interconnected time-delay systems. *J.Comput.Appl.Math.*, 193: 383–396, 2006.

D. J.Hill and J. Zhao. Global synchronization of complex dynamical networks with non-identical nodes, *Proceedings of the 47th IEEE Conference on Decision and Control*, Dec. 9-11, Cancun, Mexico, pages 817–822, 2008.

:

Controlled Consensus Criteria for a Class of Uncertain Multi-Agent Systems

Wei-Song Zhong * **Guo-Ping Liu** ** **Clive Thomas** *

* *Faculty of Advanced Technology, University of Glamorgan, Pontypridd, CF37 1DL UK (e-mail: {wzhong, ejcthoma}@glam.ac.uk).*
** *Faculty of Advanced Technology, University of Glamorgan, Pontypridd, CF37 1DL UK and NCS Lab, Central South University, 410083 PRC (e-mail: gpliu@glam.ac.uk)*

Abstract: In this paper, the consensus problem for a class of uncertain Multi-Agent Systems will be discussed. Based on the network connection topology and Lyapunov stability theory, the local and global decentralized controlled consensus problems for such systems will be investigated. Under the different assumptions, several state feedback and output feedback consensus criteria are deduced. Moreover, the assumptions adopted and decentralized control laws designed are considerably simple comparing with many existing results. An example along with the respective numerical and computer simulation results is also given to demonstrate the effectiveness of the proposed consensus control synthesis.

1. INTRODUCTION

Multi-Agent System investigations involve the study of how network architecture and interactions between network components influence global control goals. The research in this field can be categorized into two areas: One is to deal with the design of distributed estimation techniques which can be applied to the sensor networks, and the other is to deal with the control of mobile autonomous agents i.e., each agent autonomously works by using information over the network from other agents Zampieri [2008]. In both areas some important contributions have been made in recent years Desai et al. [2001], Ren et al. [2004], Yamaguchi et al. [2001], Wu et al. [2007], Porfiri et al. [2007], Cortés [2009].

The consensus problem requires the achievement of an agreement that depends on the states of all agents. The topic has been studied across many fields of science and engineering Hong et al. [2006], Xiao et al. [2008], Li et al. [2008], Kazerooni et al. [2008]. Olfati-Saber introduces two consensus criteria for networks with and without time-delays and provides convergence analysis for three kinds of Multi-Agent System with fixed and switching topologies Olfati-Saber et al. [2004]. Arack [2007] develops a passivity-based design framework to process the group coordination problem, where both fixed and time-varying communication structures are considered. Chen et al. [2009] proposes that all agents reach a consensus if a small fraction of them are controlled by simple feedback control. Tian et al [2009] investigates the robust consensus problems of second-order Multi-Agent System with diverse input delays and decentralized consensus conditions are obtained for the Multi-Agent System with symmetric coupling weights based on frequency-domain analysis. Lin et al. [2008] investigates the consensus problem for directed Multi-Agent System with external disturbances and model uncertainties for fixed and switching topologies. Bliman et al. [2008] studies the average consensus problem for undirected Multi-Agent System having communication delays and provides sufficient conditions for the existence of average consensus under bounded communication delays. Cortés [2008] characterizes a distributed algorithm that asymptotically achieved consensus and provides two discontinuous distributed algorithms that achieve max and min consensus, respectively, in finite time. Li et al. [2009] unifies the consensus in Multi-Agent System and synchronization in complex dynamical network. It's noted that the agent dynamics in most existing works are often restricted to linear and identical ones.

Inspired by these early results, the present paper will focus on the local and global consensus problems of the Multi-Agent System with uncertain coupling, and the proposed consensus property is formulated in terms of sate-feedback and output feedback. Although the behavior of the Multi-Agent System with non-identical agent dynamics is much more complicated than the identical case, this kind of Multi-Agent System may still exhibit some kinds of consensus behaviors which are far from being fully understood, and very few results have been reported as yet. Compared with many existing related results, the results of this paper have several distinct features. One is that we generalize the related results for the case of certain Multi-Agent System to the case of uncertain Multi-Agent System and the other is we design both state feedback and output feedback controllers to guarantee the uncertain Multi-Agent System achieve exact consensus. The proposed consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical agents and the conditions obtained here are easy to verify.

The rest of this paper is organized as follows. A continuous-time Multi-Agent System model with uncertain coupling is presented and some preliminaries are introduced in Section 2. The main results including state feedback and output feedback exact consensus criterion are derived in Section 3. In Section 4, two numerical simulation examples are given to verify the effectiveness of the proposed results, followed by conclusions in Section 5.

2. A CONTROLLED UNCERTAIN MULTI-AGENT SYSTEM

Consider an uncertain Multi-Agent System consisting of N identical agents with diffusive coupling, which is described by

$$\begin{aligned}\dot{x}_i(t) =& f(x_i(t),t) + g_i(x_1(t), x_2(t), \cdots, x_N(t), t) + u_i,\\ & 1 \le i \le N,\end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ represents the state variable of the i-th agent, nonlinear vector field $f(.) \in R^n$ is continuously differentiable, $g_i \in R^n$ are unknown nonlinear smooth diffusive coupling functions, $u_i \in R^n$ are the control inputs, and the coupling-control terms satisfy $g_i(s(t), s(t), \cdots, s(t), t) + u_i = 0$ for all $t \ge 0$, where $s(t)$ is a consensus solution of the isolated agent system $\dot{x}(t) = f(x(t), t)$.

Denote $s(t; t_0, x_0)$ as $s(t)$ for simplification, then $S(t) = (s^T(t), s^T(t), \cdots, s^T(t))^T$ is a consensus solution of the uncertain Multi-Agent System (1) since it is a diffusive coupling network. Here, $s(t)$ can be an equilibrium point, a periodic orbit, an aperiodic orbit, or a chaotic orbit in the phase space.

The objective of the present paper is to design controllers u_i to guarantee the uncertain Multi-Agent System (1) achieves consensus. That is, the trajectories of the closed-loop Multi-Agent Systems satisfies:

$$\lim_{t\to\infty} \|x_i(t) - s(t)\|_2 = 0,\ 1 \le i \le N. \tag{2}$$

Define the error vector by $e_i(t) = x_i(t) - s(t)$, then the error dynamical system can be given as follows:

$$\begin{aligned}\dot{e}_i(t) =& \bar{f}(x_i(t), s(t), t) + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t)\\ & + u_i, 1 \le i \le N,\end{aligned} \tag{3}$$

where $\bar{f}(x_i(t), s(t), t) = f(x_i(t), t) - f(s(t), t)$, $\bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) = g_i(x_1(t), x_2(t), \cdots, x_N(t), t) - g(s(t), s(t), \cdots, s(t), t)$.

3. MAIN RESULTS

3.1 Decentralized State Feedback Consensus

Linearizing the error system (3) around zero gives

$$\begin{aligned}\dot{e}_i(t) =& A(t)e_i(t) + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) + u_i,\\ & 1 \le i \le N,\end{aligned} \tag{4}$$

where $A(t) = Df(s(t), t)$ is the Jacobian of f evaluated at $s(t)$.

Obviously, it follows that if the pair $(A(t), I)$ is controllable, then there exist matrices $K(t), P(t) > 0, Q(t) > 0$ such that

$$\dot{P}(t) = -(A(t) + K(t))^T P(t) - P(t)(A(t) + K(t)) - Q(t). \tag{5}$$

To achieve the objective (2), the following assumptions are necessary.

Assumption 1 (A1). Suppose that there exist known first-order continuously differentiable positive definite functions $\varphi_i(.)$ with $\varphi_i(0) = 0$ and nonnegative functions $r_{ij}(t)$ such that

$$\|\bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t))\| \le \sum_{j=1}^{N} r_{ij}(t)\varphi_i(\|e_j(t)\|)$$
$$1 \le i \le N, \tag{6}$$

for $x(t) \in \Omega, t \in R^+$.

A local decentralized consensus criterion is deduced based on the above analysis.

Theorem 1. Suppose that A1 is satisfied and there exists a neighborhood about origin $\bar{\Omega} \subseteq \Omega$ such that the matrix function $W^T(e(t)) + W(e(t)) > 0$ in $\bar{\Omega}\backslash\{0\}$, then the consensus solution $S(t)$ of the uncertain Multi-Agent System (1) is locally asymptotically stable under the decentralized controllers

$$u_i = K(t)e_i(t), \quad 1 \le i \le N, \tag{7}$$

where $W(e(t)) = (w_{ij}(e(t)))_{N\times N}$ is defined by

$$w_{ij}(e(t)) = \begin{cases} \lambda_m(Q(t)) - 2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), \\ \quad i = j, \\ -2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), \quad i \ne j, \end{cases}$$

$\lambda_m(.)$ and $\lambda_M(.)$ stand for the smallest and largest eigenvalues respectively, $\phi_i(r) = \int_0^1 \frac{\partial\varphi_i(r\zeta)}{\partial\zeta} d\zeta$ with $r \in R^+$.

Proof: Substituting (7) into (4) gives the following closed-loop error system

$$\begin{aligned}\dot{e}_i(t) =& \bar{A}(t)e_i(t)\\ & + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t),\ 1 \le i \le N,\end{aligned} \tag{8}$$

where $\bar{A}(t) = (A(t) + K(t))$.

Select the following Lyapunov function candidate

$$V(e(t)) = \sum_{i=1}^{N} e_i^T(t)P(t)e_i(t), \tag{9}$$

where $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$ and $P(t)$ is defined by (5). The time derivative of $V(e(t))$ along the solution of the closed-loop error system (8) is

$$\begin{aligned}\dot{V} =& \sum_{i=1}^{N} \dot{e}_i^T(t)P(t)e_i(t) + e_i^T(t)\dot{P}(t)e_i(t) + e_i^T(t)P(t)\dot{e}_i(t)\\ =& \sum_{i=1}^{N} e_i^T(t)(\bar{A}^T(t)P(t) + P(t)\bar{A}(t) + \dot{P}(t))e_i(t)\\ & + 2\sum_{i=1}^{N} e_i^T(t)P(t)\bar{g}_i(x_1(t), x_2(t), \cdots, x_{N(t)}, s(t), t).\end{aligned} \tag{10}$$

By virtue of the conditions gives

$$\begin{aligned}&\sum_{i=1}^{N} e_i^T(t)(\bar{A}^T(t)P(t) + P(t)\bar{A}(t) + \dot{P}(t))e_i(t)\\ &\qquad \le -\sum_{i=1}^{N} e_i^T(t)Q(t)e_i(t).\end{aligned} \tag{11}$$

$$2\sum_{i=1}^{N} e_i^T(t)P(t)\bar{g}_i(x_1(t), x_2(t), \cdots, x_{N(t)}, s(t), t)$$
$$\leq 2\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|)\|e_i(t)\|\|e_j(t)\|. \tag{12}$$

Now, substituting (11) and (12) into (10) yields

$$\begin{aligned}
&\dot{V}(e(t)) \\
&\leq -\sum_{i=1}^{N} \lambda_m(Q(t))\|e_i(t)\|^2 \\
&\quad + 2\sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|)\|e_i(t)\|\|e_j(t)\| \\
&= -\frac{1}{2}(\|e_1(t)\|, \|e_2(t)\|, \cdots, \|e_N(t)\|)^T \\
&\quad (W^T(e(t)) + W(e(t)))(\|e_1(t)\|, \|e_2(t)\|, \cdots, \|e_N(t)\|).
\end{aligned}$$

From the positive definitiveness of matrix $W^T(e(t)) + W(e(t))$ in $\Omega \setminus \{0\}$, it follows that $\dot{V}(e(t))$ is a negative definite function in domain Ω. Therefore, the error dynamical system (4) is locally asymptotically stabilized by the controllers (7), i.e., $\lim_{t\to\infty} \|x_i(t) - s(t)\|_2 = 0,\ 1 \leq i \leq N$. Consequently the consensus solution $S(t)$ of uncertain Multi-Agent System (1) is locally asymptotically stable under the decentralized controllers (7). The proof is thus completed.

As a special case, assume that the nonlinear coupling terms of Multi-Agent System (1) are bounded by linear functions, that is to say, the inequalities (6) satisfy $\|\bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t)\| \leq \sum_{j=1}^{N} r_{ij}(t)\|e_j(t)\|$, then one can get the following corollary.

Corollary 1. Suppose there exists a neighborhood about the origin $\bar{\Omega} \subseteq \Omega$ such that the matrix function $W^T(e(t)) + W(e(t)) > 0$ in $\bar{\Omega} \setminus \{0\}$, then the consensus solution $S(t)$ of the Multi-Agent System (1) with linear coupling is locally asymptotically stable under the decentralized set of controllers (7), where

$$w_{ij}(e(t)) = \begin{cases} \lambda_m(Q(t)) - 2\lambda_M(P(t))r_{ij}(t), & i = j, \\ -2\lambda_M(P(t))r_{ij}(t), & i \neq j. \end{cases}$$

The following problem will focus on investigating the global decentralized consensus of the Multi-Agent System (1). Rewrite the agent dynamics $\dot{x}_i(t) = f(x_i(t), t)$ as $\dot{x}_i = A(t)x_i(t) + h(x_i(t), t)$, where $A(t) \in R^{n\times n}$ and $h : \Omega \times R^+ \to R^n$ is a smooth nonlinear function. Thus Multi-Agent System (1) is described by

$$\begin{aligned}\dot{x}_i(t) =& A(t)x_i(t) + h(x_i(t), t) \\ &+ g_i(x_1(t), x_2(t), \cdots, x_N(t), t) + u_i.\end{aligned} \tag{13}$$

Similarly, error system can be gotten as follows

$$\begin{aligned}\dot{e}_i(t) =& A(t)e_i(t) + \bar{f}(x_i(t), s(t), t) \\ &+ \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) + u_i,\end{aligned} \tag{14}$$

where $\bar{f}(x_i(t), s(t), t) = h(x_i(t), t) - h(s(t), t)$.

Assumption 2 (A2). Suppose that there exist known first-order continuously differentiable positive definite functions $\gamma_i(.)$ with $\gamma_i(0) = 0$ such that $\|\bar{f}(x_i(t), s(t), t)\| \leq \gamma_i(\|e_i(t)\|)$.

Assumption 3 (A3). Suppose that there exist a neighborhood about origin $\bar{\Omega} \subseteq \Omega$ such that the matrix function $W^T(e(t)) + W(e(t)) > 0$ in $\bar{\Omega}\setminus\{0\}$, where $W(e(t)) = (w_{ij}(e(t)))_{N\times N}$ is defined by

$$w_{ij}(e(t)) = \begin{cases} \lambda_m(Q(t)) - 2\lambda_M(P(t))\kappa_i(\|e_i(t)\|) \\ \qquad - \lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i = j, \\ -\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i \neq j, \end{cases}$$

for $i, j = 1, 2, \cdots, N$, $\phi_i(r) = \int_0^1 \frac{\partial\varphi_i(r\zeta)}{\partial\zeta}d\zeta$ and $\kappa_i(r) = \int_0^1 \frac{\partial\gamma_i(r\zeta)}{\partial\zeta}d\zeta$ with $r \in R^+$.

On the grounds of A1, A2 and A3, the following global decentralized consensus criterion can be gotten easily.

Theorem 2. Suppose that A1, A2 and A3 are satisfied. Then the consensus solution $S(t)$ of the uncertain Multi-Agent System (1) is globally asymptotically stable under the decentralized controllers (7).

The proof is very similar to that of the Theorem 1, and is omitted here.

Corollary 2. Suppose A2 is satisfied and $\bar{f}(x_i(t), s(t), t) = 0$ for $e_i(t) \in \Xi$ with $\Xi = \{(e_i(t), t)|P(t)e_i(t) = 0, t \in R^+\}$ for $1 \leq i \leq N$. Then the consensus solution $S(t)$ of the uncertain Multi-Agent System (1) is globally asymptotically stable under the decentralized controllers

$$u_i = K(t)e_i(t) + \rho(e_i(t)), \quad 1 \leq i \leq N, \tag{15}$$

where $\rho(.)$ is given by

$$\rho(e_i(t) = \begin{cases} -\dfrac{P(t)e_i(t)}{\|P(t)e_i(t)\|^2}\lambda_M(P(t))\gamma_i(\|e_i(t)\|)\|e_i(t)\|, \\ \qquad\qquad P(t)e_i(t) \neq 0, \\ 0, \qquad\quad P(t)e_i(t) = 0. \end{cases} \tag{16}$$

The proof is very similar to that of the Theorem 1, and is omitted here.

3.2 Decentralized Output Feedback Consensus

The next, controlled uncertain Multi-Agent System with outputs will be considered.

$$\begin{aligned}\dot{x}_i(t) &= A(t)x_i(t) + h(x_i(t), t) \\ &\quad + g_i(x_1(t), x_2(t), \cdots, x_N(t), t) + u_i, \\ y_i(t) &= H(t)x_i(t),\ i = 1, 2, \cdots, N,\end{aligned} \tag{17}$$

where $y_i(t) \in R^n$ is the output of the $i-$th node, $H(t) \in R^{n\times n}$ and other statements on the Multi-Agent System are the same as in (1) and (13).

Similarly, the following error system can be obtained easily

$$\begin{aligned}\dot{e}_i(t) &= A(t)e_i(t) + \bar{f}(x_i(t), s(t), t) \\ &\quad + \bar{g}_i(x_1(t), x_2(t), \cdots, x_N(t), s(t), t) + u_i, \\ \bar{y}_i(t) &= H(t)e_i(t) + H(t)s(t),\ i = 1, 2, \cdots, N,\end{aligned} \tag{18}$$

where $\bar{f}(x_i(t), s(t), t) = h(x_i(t), t) - h(s(t), t)$.

Then the object is to achieve consensus of the Multi-Agent System (17) by designing decentralized output feedback controllers $u_i(y_i(t))$, that is, the trajectories of the closed-loop systems satisfy (2). According to (18), the consensus problem of the Multi-Agent System (17) is equivalent to the stabilization problem of the error dynamical system (18). To achieve the objective, the following assumptions are needed.

Assumption 4 (A4). Suppose that there exist known first-order continuously differentiable positive definite functions $r_i(.)$ with $r_i(0) = 0$ such that $\|\bar{f}(x_i(t), s(t), t)\| \leq \bar{\gamma}_i(t)\|\bar{y}_i(t)\|$.

Assumption 5 (A5). Suppose there exists a neighborhood about origin $\bar{\Omega} \subseteq \Omega$ such that the matrix function $W^T(e(t)) + W(e(t)) > 0$ in $\bar{\Omega}\backslash\{0\}$, where $W(e(t)) = (w_{ij}(e(t)))_{N\times N}$ is defined by

$$w_{ij}(e) = \begin{cases} \lambda_m(Q(t)) - \epsilon_1 \\ -\dfrac{1}{\epsilon_2}\|(K^T(t)D(t) - \dfrac{\bar{\gamma}_i^2(t)}{2\epsilon_1}\|P(t)\|^2 I_n)\|^2\|H(t)\|^2 \\ \quad - 2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i = j, \\ -2\lambda_M(P(t))r_{ij}(t)\phi_i(\|e_j(t)\|), & i \neq j, \end{cases}$$

for $r_{ij}(t)$, $\lambda_m(.)$ and $\lambda_M(.)$ stand for the smallest and largest eigenvalues respectively, $\phi_i(r) = \int_0^1 \frac{\partial\varphi_i(r\zeta)}{\partial\zeta}d\zeta$ with $r \in R^+$.

Theorem 3. Suppose that A1, A4 and A5 are satisfied and there exist matrix $K(t) \in R^{n\times n}$, nonsingular matrix $D(t) \in R^{n\times n}$, two symmetric positive definite matrices $P(t) \in R^{n\times n}$, $Q(t) \in R^{n\times n}$ and a positive constant ϵ_1 such that $(A(t) + K(t)H(t))^T P(t) + P(t)(A(t) + K(t)H(t)) + \dot{P}(t) = -Q(t)$, $P(t) = D(t)H(t)$ and $\epsilon_2 \geq \frac{\bar{\gamma}_i^2(t)}{\epsilon_1}\|P(t)\|^2$. Then the consensus solution $S(t)$ of the uncertain Multi-Agent System (17) is globally asymptotically stable under the decentralized output feedback controllers

$$u_i(\bar{y}_i(t)) = K(t)\bar{y}_i(t) + u_i^a(\bar{y}_i(t)), \tag{19}$$

where $u_i^a(\bar{y}_i(t)) = -\frac{\bar{\gamma}_i^2(t)}{2\epsilon_1}(D^T(t))^{-1}\|P(t)\|^2\bar{y}_i(t)$. The proof is similar to that of the Theorem 2, so is omitted here.

Corollary 3. Suppose that all time-varying parameters in Theorem 3 are invariable for $t > 0$ and function matrix $W^T(e(t)) + W(e(t))$ is positive definite in $\bar{\Omega}\backslash\{0\}$ with $W(e(t)) = (w_{ij}(e(t)))_{N\times N}$. Then the consensus solution $S(t)$ of the uncertain Multi-Agent System (17) is globally asymptotically stable under the decentralized output feedback controllers $u_i(\bar{y}_i(t)) = (K - \frac{\bar{\gamma}_i^2}{2\epsilon_1}(D^T)^{-1}\lambda_M^2(P))\bar{y}_i(t)$, where

$$w_{ij}(e(t)) = \begin{cases} \lambda_m(Q) - \epsilon_1 \\ \quad -\dfrac{1}{\epsilon_2}\|K^T D - \dfrac{\bar{r}_i^2}{2\epsilon_1}\lambda_M^2(P)I_n\|^2\lambda_M(H) \\ \quad - 2\lambda_M(P)r_{ij}\phi_i(\|e_j(t)\|), & i = j, \\ -2\lambda_M(P)r_{ij}\phi_i(\|e_j(t)\|), & i \neq j, \end{cases}$$

$K, D, P, Q, \epsilon_1, \epsilon_2, \gamma_i$ and r_{ij} are corresponding constant parameters in the Theorem 3.

4. EXAMPLES

In this section, several numerical simulations for verifying the effectiveness of the proposed consensus criteria will be given.

Example 1. Consider the Multi-Agent System consisting of 2 identical second-order nodes, which is described by

$$\begin{aligned} \begin{pmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{pmatrix} &= \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} + \begin{pmatrix} sin^2(x_{11})x_{12} \\ cos(x_{12})x_{11} \end{pmatrix} + u_1 \\ &+ 0.05sin(t)e^{-t}\begin{pmatrix} cos^2(x_{12}) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} \\ &+ sin(t)e^{-2t}\begin{pmatrix} sin(x_{12}) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}, \\ \begin{pmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{pmatrix} &= \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} + \begin{pmatrix} sin^2(x_{21})x_{22} \\ cos(x_{22})x_{21} \end{pmatrix} + u_2 \\ &+ e^{-t}\begin{pmatrix} 1 & 0 \\ 0 & sin(x_{21}^2) \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} \\ &+ sin(t)e^{-2t}\begin{pmatrix} cos(x_{12}) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}. \end{aligned}$$

In the simulation, the controller gain matrix and other parameter matrix are chosen as follows

$$K(t) = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}, \quad P(t) = \begin{pmatrix} 0.6406 & 0.2031 \\ 0.2031 & 0.5156 \end{pmatrix}.$$

Computing directly, we have that $Q(t) = diag\{6, 6\}$ and

$$W(t) = \begin{pmatrix} 4.42 - 0.04|sint|e^{-t} & -0.79|cos(t)|e^{-2t} \\ -0.79e^{-t} & 4.42 - 0.79|sin(t)|e^{-2t} \end{pmatrix},$$

$\|\bar{g}_1\|_2 \leq \|e_1\|_2$, $\|\bar{g}_2\|_2 \leq \|e_2\|_2$, $\|\bar{h}_1\|_2 \leq 0.05e^{-t}|sin(t)| \|e_1\|_2 + e^{-2t}|cos(t)|\|e_2\|_2$, and $\|\bar{h}_2\|_2 \leq e^{-t}\|e_1\|_2 + e^{-2t}|sin(t)|\|e_2\|_2$.

It is observed that the conditions of the Theorem 2 are satisfied. Choose initial state as $x_0 = (-0.5, 0.7, 0.45, -0.3)$ and the consensus error e_i and the corresponding control signal are shown in Figure 1.

Example 2: Consider the following controlled time-varying Multi-Agent System consisting of 2 identical second-order nodes.

$$\begin{aligned} \begin{pmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{pmatrix} &= \begin{pmatrix} -x_{12} \\ -sin(-x_{11} + 2x_{12}) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1 \\ &- 0.04sin(t)\begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} \\ &+ 0.04sin(t)\begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}, \\ y_1 &= x_{11} - 2x_{12}, \\ \begin{pmatrix} \dot{x}_{21} \\ \dot{x}_{22} \end{pmatrix} &= \begin{pmatrix} -x_{22} \\ -sin(-x_{21} + 2x_{22}) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_2 \\ &+ 0.03sin(t)\begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} \\ &- 0.03sin(t)\begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}, \\ y_2 &= x_{21} - 2x_{22}. \end{aligned}$$

Let $u_1 = sin(-x_{11} + 2x_{12}) + x_{11} - 2x_{12} = sin(y_1) - y_1$, $u_2 = sin(-x_{21} + 2x_{22}) - x_{21} + 2x_{22} = sin(y_2) - y_2$, $V(z_1, z_2) = 3z_1^2 - 2z_1z_2 + z_2^2$. A direct computation gives: $r_{11}(\tau) = r_{21}(\tau) = (2 - \sqrt{2})\tau^2$, $r_{21}(\tau) = r_{22}(\tau) = (2 - \sqrt{2})\tau^2$, $k_1 = 3$, $k_2 = 7$,

$$W(t) = \begin{pmatrix} 3 - 0.28|sin(t)|e^{-t} & 0.28|sin(t)|e^{-t} \\ 0.21|cos(t)|e^{-t} & 3 - 0.21|cos(t)|e^{-t} \end{pmatrix}$$

is positive definite for any $t > 0$. The conditions of the Theorem 3 are satisfied. Therefore, the above agent systems

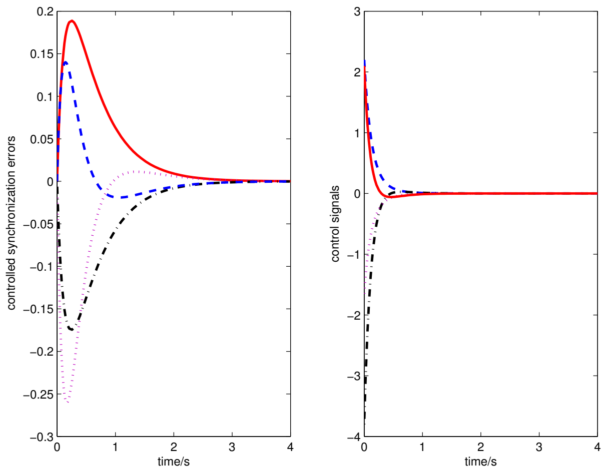

Fig. 1. Consensus errors and control signals.

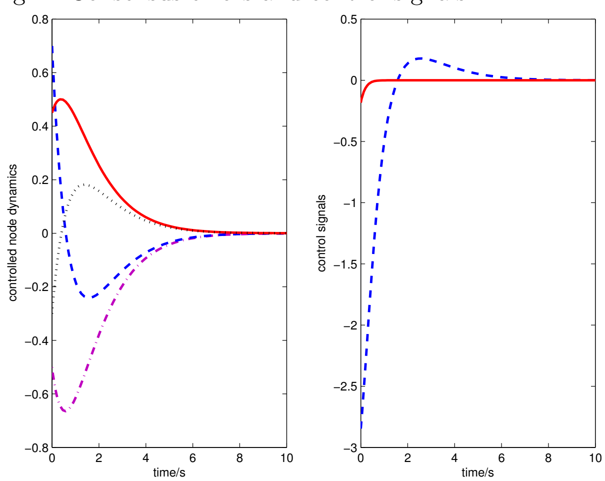

Fig. 2. Consensus errors and control signals.

can be stabilized by the decentralized output feedback controller. Choose initial state as $x^0 = (-0.5, 0.7, 0.45, -0.3)$ and the simulation results are depicted in Figure 2.

5. CONCLUSIONS

In this paper, a representation model for class of controlled time-varying Multi-Agent System with uncertain coupling term was proposed and decentralized state feedback controllers and decentralized output feedback controllers were designed to asymptotically stabilize the system. Several network consensus criteria have been proved by using Lyapunov stability theory. Compared with all relevant previous results, the results are rather general and simpler and many previous results can be viewed as the special cases of the present results. Finally, numerical simulation demonstrated the effectiveness of the proposed results.

REFERENCES

S. Zampieri. Trends in networked control systems. *Proceedings of 17th World Congress of IFAC*, July 6-11, Seoul, Korea, pages 2886–2894, 2008.

J.P. Desai, J.P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust. *IEEE Int. Trans. on Robotics and Automation*, 17(6): 905–908, 2001.

W. Ren and R.W. Beard. Formation feedback control for multiple spacecraft via virtual structures. *IEE Proc. Control Theory and Applications*, 151(3): 357–368, 2004.

H. Yamaguchi, T. Arai and G. Beni. A. distributed control scheme for multiple robotic vehicles to make group formations. *Robitics and Autonomous Systems*, 36: 125–147, 2001.

Z.P. Wu, Z.H. Guan, T. Li. Group motion control of multi-agent systems based on complex network. *Proceedings of the 26th Chinese Control Conference*, July 26-31, Hunan, China, pages 63–67, 2007.

M. Porfiri, D.G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach. *Automatica*, 43: 1318-1328, 2007.

J. Cortés. Global formation-shape stabilization of relative sensing networks. *2009 American Control Systems*, June 10-12, Hyatt Regency Riverfront, St. Louis, USA, pages 1460–1465, 2009.

Y.G. Hong, J.P. Hu and L.X. Gao. Tracking control for multi-agent consensus with an active leader and variable topology. *Automatica*, 42: 1177–1182, 2006.

F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Transactions on Automatic Control*, 53(8): 1804–1816, 2008.

T. Li and J.F. Zhang. Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash equilibria. *Automatica*, 44: 713–725, 2008.

E.S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information. *Automatica*, 44: 2766–2777, 2008.

R. Olfati-Saber and R.M. Murray. Consensus problmes in networks with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9): 1520–1533, 2004.

M. Arack. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 52(8): 1380–1390, 2007.

F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu and Z.Z. Yuan. Reaching a consensus via pinning control. *Automatica*, 45: 1215–1220, 2009.

Y.P. Tian and C.L Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations. *Automatica*, 45: 1347–1353, 2009.

P. Lin, Y.M. Jia and L. Li. Distributed robust H_∞ Consensus control in directed networks of agents with time-delay. *Systems and Control Letters*, 57(8): 643–653, 2008.

P.A. Bliman and G.F. Trecate. Average consensus problems in networks of agents with delayed communications. *Automatica*, 44: 1985–1995, 2008.

J. Cortés. Distributed algorithms for reaching consensus on general functions. *Automatica*, 44(3): 726–737, 2008.

Z.K. Li, Z.S. Duan, G.R. Chen and L. Huang. Consensus of multi-agent systems and synchronization of complex networks: a unified viewpoint. *IEEE Transactions on Circuits and Systems-I*, 57(1): 213–224, 2009.

:

Controlled Consensus of Multi-Agent Systems with Communication Time Delay

Wei-Song Zhong, Guo-Ping Liu and Clive Thomas

***Abstract*— This paper investigates the global exact consensus problem of networked Multi-Agent Systems (MAS) consisting of nonlinear, identical agent dynamics and communication time-delay topology. We derive global exact consensus conditions by using of linear feedback and apaptive feedback respectively, and the proposed results are theoretically proved to be effective based on the Lyapunov-Krasovskii functional method. The proposed consensus criteria ensure that all agents eventually move along the desired dynamic trajectory which is derived from the average dynamics of all agents here, meanwhile, the proposed consensus criteria generalize many existing results and can be viewed as an extension of such relevant results. The effectiveness of the results is also demonstrated through a numerical simulation finally.**

I. Introduction

MULTI-AGENT SYSTEMS (MAS) deals with the study of how network architecture and interactions between network components influence global control goals. The research in this field can be categorized into two areas. One area is to deal with the design of distributed estimation techniques which can be applied to the sensor networks, and the other area is with the control of mobile autonomous agents i.e., each agent autonomously works by using information over the network from other agents [1]. In both areas some important contributions have been made in recent years [2], [3], [4], [5].

The consensus problem means to reach an agreement that depends on the state of all agents. The topic has been studied across many fields of science and engineering [6], [7], [8], [9], [10]. Reza introduces two consensus criteria for networks with and without time-delays and provides convergence analysis for three kinds of MAS with fixed and switching topologies [11]. [12] develops a passivity-based design framework to process the group coordination problem, where both fixed and time-varying communication structures are considered. [13] proposes that all agents reach a consensus if a small fraction of them are controlled by simple feedback control. [14] investigates the robust consensus problem of second-order MAS with diverse input delays and decentralized consensus conditions are obtained for the MAS with symmetric coupling weights based on frequency-domain analysis. [15] investigates the consensus problem for directed MAS with external disturbances and model uncertainties for fixed and switching topologies. [16] studies the average consensus problem for undirected MAS having communication delays and provides sufficient conditions for the existence of average consensus under bounded communication delays. [17] characterizes a distributed algorithm that asymptotically achieves consensus and provides two discontinuous distributed algorithms that achieve max and min consensus, respectively, in finite time. It's noted that the agent dynamics in most existing literatures are often restricted to linear and identical ones, especially for the single integrator dynamics and double integrator dynamics, obviously, in practice, this is not always the case. The controlled consensus problem of MAS with nonlinear agent dynamics and communication delay are more complicated and just a few results have been made [18], [19], [20], [21], [22], [23]. In addition, most research in consensus problems usually assume that the final consensus value to be reached is inherently constant, which may not be the case in the sense that the information state of each agent may be dynamically evolving in time according to some inherent dynamics. It is interesting to study consensus problems where the final consensus value evolves with time or as a function of environmental dynamics.

The present paper will focus on the global consensus problems of MAS with nonlinear agent dynamics and communication delay. Compared with many existing results, this paper make two significant advances. One is that we choose the average agent dynamics as the desired moving trajectories instead of a constant, and the other is we introduce linear feedback and adaptive feedback control to guarantee exact consensus of the MAS respectively.

The rest of this paper is organized as follows. A controlled continuous-time MAS model with nonlinear agent dynamics and communication time-delay is presented and some preliminaries are introduced in Section II. The main results including linear feedback control and adaptive feedback control exact consensus criteria are derived in Section III. In Section IV, a numerical simulation example is given to verify the effectiveness of the proposed results, followed by conclusions in Section V.

II. Preliminaries

Let $G=(\mathcal{V},\mathcal{A})$ be a graph of order N consists of a set of vertices $\mathcal{V}=\{v_1,v_2,\cdots,v_N\}$ and a set of edges $\mathcal{A}\subseteq\mathcal{V}\times\mathcal{V}$. An edge (v_j,v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i=\{v_j\in\mathcal{V}:(v_j,v_i)\in\mathcal{A}\}$.

We consider a MAS consisting of N agents with commu-

Wei-Song Zhong, Guo-Ping Liu and Clive Thomas are with the Faculty of Advanced Technology, University of Glamorgan, Pontypridd, UK and Guo-Ping Liu is also with CTGT Centre, Harbin Institute of Technology, Harbin, China(email: {wzhong,gpliu,ejcthoma}@glam.ac.uk).

nication delay:

$$\dot{x}_i = f(x_i) + \sum_{j\in\mathcal{N}_i}^{N} c_{ij}\Gamma x_j(t-\tau) + u_i, i = 1, 2, \cdots, N, \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f(x_i(t))$ is a continuously differentiable vector function, $u_i \in R^n$ is a local controller to be designed, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix. The adjacency matrix $C = (c_{ij}) \in R^{N\times N}$, representing the communication topology relation of the MAS, is symmetric and irreducible, $c_{ij} \geq 0$ and $c_{ii} = -\sum_{j\neq i} c_{ij}$. $\tau > 0$ is a constant time delay.

The desired moving trajectory is chosen as

$$\bar{x}(t) = \frac{1}{N}\sum_{k=1}^{N} x_i(t). \quad (2)$$

The consensus problem is solvable if the states of all agents satisfy $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$, $\forall i, j = 1, 2, \cdots, N$.

Define the error vector $e_i(t) = x_i(t) - \bar{x}(t)$, together with the MAS (1) results in the error system in terms of $e_i(t)$:

$$\begin{aligned}\dot{e}_i(t) = f(x_i(t)) - f(\bar{x}(t)) + \sum_{j=1}^{N} c_{ij}\Gamma e_j(t-\tau) \\ + f(\bar{x}(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t)) + u_i.\end{aligned} \quad (3)$$

It needs to say that the exact consensus problem of the MAS (1) and the stabilization problem of the error system (3) are equivalent to each other.

Assumption 1 Suppose there exists a positive constant L such that

$$\|f(x_i(t)) - f(\bar{x}(t))\| \leq L\|e_i(t)\| \quad (4)$$

holds for $i = 1, 2, \cdots, N$.

III. Main Results

A. Global Consensus via Liner Feedback Control

Theorem 1. Suppose Assumption 1 holds and then the MAS (1) achieves global exact consensus under the set of controllers

$$u_i = k_i e_i(t), \quad i = 1, 2, \cdots, N, \quad (5)$$

where $-k_i - L - \lambda + c_{ii}(\|\Gamma^2\| + 1) < 0$ and $\lambda > 0$.

Proof: Choose the Lyapunov functional candidate as

$$V(t) = \sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{T} c_{ii}\int_{t-\tau}^{t} e_i^T(\theta)\Gamma^2 e_i(\theta)d\theta,$$

then the time derivative of $V(t)$ along the solution of the error system (3) is given as follows

$$\begin{aligned}\dot{V}(t) = &\sum_{i=1}^{N} e_i^T(t)[f(x_i(t)) - f(\bar{x}(t))] \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}e_i^T(t)\Gamma e_j(t-\tau) \\ &+ \sum_{i=1}^{N} e_i^T(t)(f(\bar{x}(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t))) \\ &+ \sum_{i=1}^{N} e_i^T(t)u_i - \sum_{i=1}^{N} c_{ii}e_i^T(t)\Gamma^2 e_i(t) \\ &+ \sum_{i=1}^{N} c_{ii}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau).\end{aligned}$$

Since $\sum_{i=1}^{N} e_i^T(t)(f(\bar{x}(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t))) = 0$, together with the condition (4) and the controllers (5), then we have

$$\begin{aligned}\dot{V}(t) \leq &\sum_{i=1}^{N}(-\lambda + c_{ii})e_i^T(t)e_i(t) \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}e_i^T(t)\Gamma e_j(t-\tau) + \sum_{i=1}^{N} c_{ii}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau) \\ \leq &-\lambda\sum_{i=1}^{N} e_i^T(t)e_i(t) + \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ji}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau) \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}e_i^T(t)\Gamma e_j(t-\tau) - \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ij}e_i^T(t)e_i(t)\end{aligned}$$

According to the symmetry of adjacency matrix C, we can obtain $\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ji}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau) = \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} c_{ij}e_j^T(t-\tau)\Gamma^2 e_j(t-\tau)$, then we have the differential coefficient of V as

$$\begin{aligned}\dot{V}(t) \leq &-\lambda\sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} |c_{ij}|\|e_i(t) - \frac{\Gamma}{2}e_j(t-\tau)\|^2 \\ &- \frac{3}{4}\sum_{i=1}^{N}\sum_{j=1}^{N} |c_{ij}|e_i^T(t-\tau)\Gamma^2 e_i(t-\tau),\end{aligned}$$

thus we have

$$\dot{V}(t) \leq -\lambda\sum_{i=1}^{N} e_i^T(t)e_i(t),$$

in consequence, the MAS(1) achieves global exact consensus under the controllers (5). This completes the proof.

B. Global Consensus via Adaptive Feedback Control

Theorem 2. Suppose Assumption 1 holds and then the MAS (1) achieves global exact consensus under the set of controllers

$$\begin{cases} u_i = k_i(t)e_i(t), \\ \dot{k}_i(t) = h_i e_i^T(t)e_i(t), \ h_i > 0, \quad i = 1, 2, \cdots, N, \end{cases} \quad (6)$$

where $L - h_i + c_{ii}(\|\Gamma^2\| + 1) + \lambda < 0$ and $\lambda > 0$.

Proof: Choose the Lyapunov functional candidate as

$$V(t) = \sum_{i=1}^{N} e_i^T(t)e_i(t) - \sum_{i=1}^{T} c_{ii} \int_{t-\tau}^{t} e_i^T(\theta)\Gamma^2 e_i(\theta)d\theta + \frac{1}{2}\sum_{i=1}^{N} \frac{(k_i(t) - h_i)^2}{h_i}.$$

The derivative of V with respect to time t along the trajectories of (3) is then given by

$$\begin{aligned}\dot{V}(t) = &\sum_{i=1}^{N} e_i^T(t)[f(x_i(t)) - f(\bar{x}(t))] \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} e_i^T(t)\Gamma e_j(t-\tau) \\ &+ \sum_{i=1}^{N} e_i^T(t)(f(\bar{x}(t)) - \frac{1}{N}\sum_{i=1}^{N} f(x_i(t))) \\ &+ \sum_{i=1}^{N} e_i^T(t)u_i - \sum_{i=1}^{N} c_{ii}e_i^T(t)\Gamma^2 e_i(t) \\ &+ \sum_{i=1}^{N} c_{ii}e_i^T(t-\tau)\Gamma^2 e_i(t-\tau) + \sum_{i=1}^{N} \frac{k_i(t) - h_i}{h_i}\dot{k}_i(t).\end{aligned}$$

The remainder proof is very similar to that of the Theorem 1, so is omitted here.

At last, we still have

$$\dot{V}(t) \leq -\lambda \sum_{i=1}^{N} e_i^T(t)e_i(t),$$

which means that the MAS(1) achieves global exact consensus under the controllers (6). This completes the proof.

IV. Simulation Study

Consider a MAS constructed with 11 Lorenz chaotic systems and each agent dynamic is given by

$$\begin{cases}\dot{x}_1(t) = -10x_1(t) + 10x_2(t) \\ \dot{x}_2(t) = 28x_1(t) - x_2(t) - x_1(t)x_3(t) \\ \dot{x}_3(t) = -\frac{8}{3}x_3(t) + x_1(t)x_2(t)\end{cases}$$

The communication coupling matrix $C = (C_1^T C_2^T \cdots C_{11}^T)$, $C_1 = (-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1)$, $C_2 = (1\ {-8}\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1)$, $C_3 = (1\ 1\ {-6}\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1)$, $C_4 = (0\ 1\ 1\ {-5}\ 0\ 1\ 1\ 0\ 0\ 1\ 0)$, $C_5 = (1\ 1\ 0\ 0\ {-6}\ 0\ 1\ 1\ 1\ 1\ 0)$, $C_6 = (1\ 0\ 0\ 1\ 0\ {-5}\ 1\ 0\ 1\ 1\ 0)$, $C_7 = (1\ 1\ 0\ 1\ 1\ 1\ {-7}\ 1\ 0\ 1\ 0)$, $C_8 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ {-5}\ 0\ 1\ 1)$, $C_9 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ {-6}\ 1\ 1)$, $C_{10} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ {-10}\ 1)$, $C_{11} = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ {-6})$. $\Gamma = diag\{3, 3, 3\}$, $\bar{x}(t) = \frac{1}{11}\sum_{i=1}^{N} x_i(t)$, $\lambda = 1$, $\tau = 0.4$. The Lorenz system is a bounded chaotic attractor, i.e., there exists a constant M satisfying$\|f(x_i(t)) - f(\bar{x}(t))\| \leq 2M\|e_i(t)\|$. $x_0 = 20 * (1, 0.5, -1, 1.2, 0.6, -1.2, 1.4, 0.7, -1.4, 1.6, 0.8, -1.6, 1.8, 0.9, -1.8, 2, 1, -2, -1.8, 1.1, 1.8, -1.6, 1.2, 1.6, -1.4, 1.3, 1.4, -1.2, 1.4, 1.2, -1, 1.5, 1, -3, -2.5, -2, -1.5, -1, -0.5, -0.5, -1, -1.5, -2, -2.5)^T$. Adaptive gains $(h_1, h_2, \cdots, h_{11}) = (0.3, 0.4, 0.5, 0.6, 0.3, 0.4, 0.5, 0.6, 0.3, 0.4, 0.5)$. Applying Theorem 2 we know that the exact consensus is achived. Simulation results are depicted in Figure 1 to Figure 5 respectively

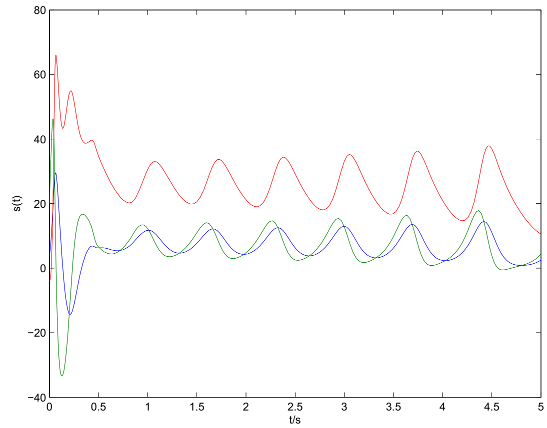

Fig. 1. Desired Moving Trajectories

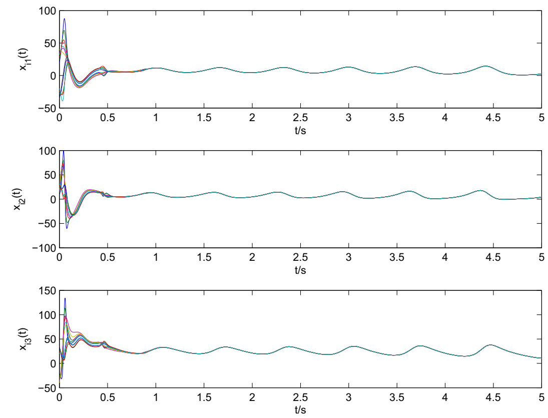

Fig. 2. Agent Dynamics

V. Conclusion

We have studied the exact consensus problem for a MAS with nonlinear agent dynamics and communication delay. We use the average dynamics of all agents as the desired moving trajectories, then we have presented linear feedback and adaptive feedback to guarantee its global exact consensus based on the Lyapunov stability theory. The controllers are very simple in form, but are more effective to resolve the consensus problem of the MAS with nonlinear agent dynamics and communication delay. The simulation results have illustrated the effectiveness of the proposed results. It should be noted that the conditions are still restrictive and all the delays are the same, further work regarding this topic we'll focus on these problems.

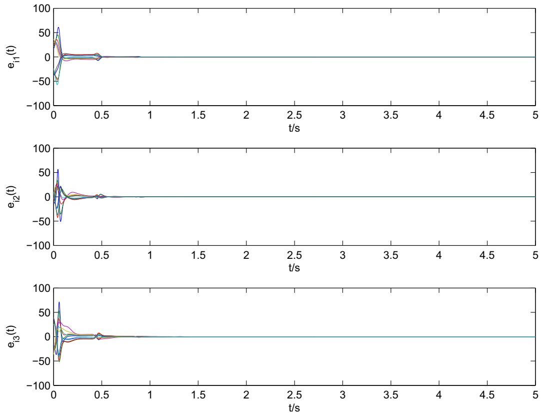

Fig. 3. Error System Dynamics

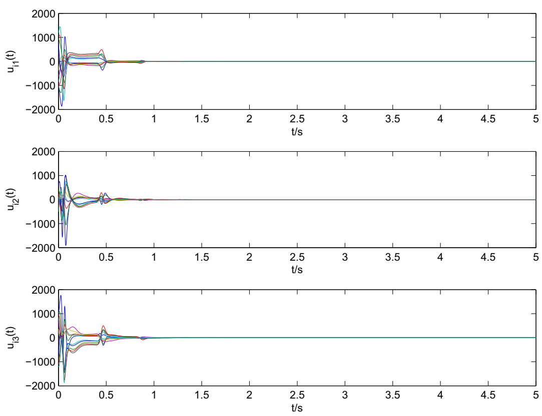

Fig. 4. Adaptive Controllers

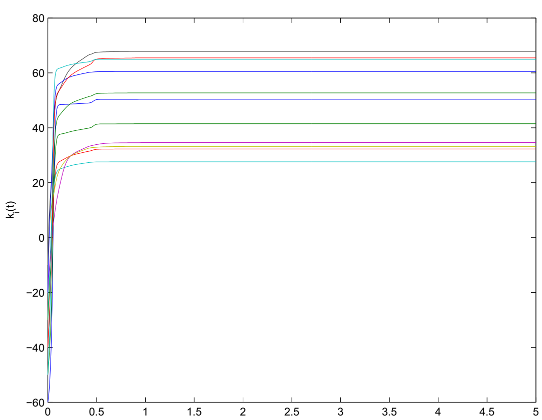

Fig. 5. Adaptive Gain Curves

REFERENCES

[1] Sandro Zampieri. Trends in networked control systems[C], *Proceedings of 17th World Congress of IFAC*, Seoul, Korea, July 6-11, 2008, pp. 2886 – 2894.

[2] J. P. Desai, J. P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust[J]. *IEEE Int. Trans. on Robotics and Automation*, 2001, 17(6): 905 – 908.

[3] H. Yamaguchi, T. Arai and G. Beni. A distributed control scheme for multiple robotic vehicles to make group formations[J]. *Robitics and Autonomous Systems*, 2001, 36: 125-147.

[4] Z. P. Wu, Z. H. Guan, T. Li. Group motion control of multi-agent systems based on complex network[C], *Proceedings of the 26th Chinese Control Conference*, Hunan, China, July 26-31, 2007, pp: 63 – 67.

[5] M. Porfiri, D. G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach[J]. *Automatica*, 2007, 43: 1318 – 1328.

[6] Y. Liu, Y. M. Jia, J. P. Du and S. Y. Yuan. Dynamic output feedback control for consensus of multi-agent systems: an H_∞ approach[C]. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 4470 – 4475.

[7] Y. G. Hong, J. P. Hu and L. X. Gao. Tracking control for multi-agent consensus with an active leader and variable topology[J]. *Automatica*, 2006, 42: 1177 – 1182.

[8] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays[J]. *IEEE Transactions on Automatic Control*, 2008, 53(8): 1804 – 1816.

[9] T. Li and J. F. Zhang. Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash equilibria[J]. *Automatica*, 2008, 44: 713 – 725.

[10] E. S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information[J]. *Automatica*, 2008, 44: 2766 – 2777.

[11] R. Olfati-Saber and R. M. Murray. Consensus problmes in networks with switching topology and time-delays[J]. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1520 – 1533.

[12] M. Arack. Passivity as a design tool for group coordination[J]. *IEEE Transactions on Automatic Control*, 2007, 52(8): 1380 – 1390.

[13] F. Chen, Z. Q. Chen, L. Y. Xiang, Z. X. Liu and Z. Z. Yuan. Reaching a consensus via pinning control[J]. *Automatica*, 2009, 45: 1215 – 1220.

[14] Y. P. Tian and C. L. Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations[J]. *Automatica*, 2009, 45: 1347 – 1353.

[15] P. Lin, Y. M. Jia and L. Li. Distributed robust H_∞ Consensus control in directed networks of agents with time-delay[J]. *Systems and Control Letters*, 2008, 57(8): 643 – 653.

[16] P. A. Bliman and G. F. Trecate. Average consensus problems in networks of agents with delayed communications[J]. *Automatica*, 2008, 44: 1985 – 1995.

[17] J. Cortes. Distributed algorithms for reaching consensus on general functions[J]. *Automatica*, 2008, 44(3): 726 – 737.

[18] Z. G. Hou, L. Chen, M. Tan. Decentralized robust adaptive control for the multiagent system consensus problem using neural networks[J]. *IEEE Transactions on Systems, Man, and Cybgernetics-Part B: Cybernetics*, 2009, 39(3): 636 – 647.

[19] Z. K. Li, Z. S. Duan, G. R. Chen and L. Huang. Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint[J]. *IEEE Transactions on Circuits and Systems-I*, 2009, 57(1): 213 – 224.

[20] D. J.Hill and J. Zhao. Global synchronization of complex dynamical networks with non-identical nodes[C], *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 9-11, 2008, 817 – 822.

[21] J. Xiang and G.R. Chen. On the V-stability of complex dynamical networks[J]. *Automatica*, 2007, 43: 1049 – 1057.

[22] C.C. Hua, X.P. Guan and P. Shi. Decentralized robust model reference adaptive cotnrol for interconnected time-delay systems[J]. *J.Comput.Appl.Math.*, 2006, 193: 383 – 396.

[23] R.K. Upadhyay, V. Rai. Complex dynamics and synchronization in two non-identical chaotic ecological systems[J]. *Chaos, Solutions and Fractals*, 2009, 40(5): 2233 – 2241.

:

Controlled Consensus of Multi-Agent Systems with Different Agent Dynamics

Wei-Song Zhong * **Guo-Ping Liu** ** **David Rees** ***

* *University of Glamorgan, Pontypridd, UK, (e-mail: wzhong@glam.ac.uk).*
** *University of Glamorgan, Pontypridd, UK; The CSIS Laboratory, Chinese Academy of Sciences, Beijing, China, (e-mail: gpliu@glam.ac.uk)*
*** *University of Glamorgan, Pontypridd, UK, (e-mail: drees@glam.ac.uk)*

Abstract: The problem of consensus on Multi-Agent System of structurally different dynamical agent is investigated. Consensus of Multi-Agent System is usually defined in terms of identical accordance on the evolution of each individual agent in the network. However, for a network consisting of strictly different agents, this type of consensus should be redefined. In this case, a generalized definition of consensus can be considered, where the evolution of each agent can be related to others in terms of a map. In order to achieve consensus on a network of strictly different agents, local linear and nonlinear controllers are designed which force the network to achieve bounded consensus or exact consensus respectively. The main results of this study are illustrated by numerical simulations.

Keywords: Consensus, Multi-Agent Systems, Different Agent Dynamics, Nonlinear Controllers, Error System, Stabilization.

1. INTRODUCTION

Multi-Agent Systems (MAS) deals with the study of how network architecture and interactions between network components influence global control goals. The research in this field can be categorized into two areas: One is to deal with the design of distributed estimation techniques which can be applied to the sensor networks, and the other is to deal with the control of mobile autonomous agents i.e., each agent autonomously works by using information over the network from other agents(1). In both areas some important contributions have been made in recent years(2; 3; 4; 5; 6; 7).

The consensus problem means to reach an agreement that depends on the states of all agents. The topic has been studied across many fields of science and engineering(8; 9; 10; 11; 12). Reza introduces two consensus criteria for networks with and without time-delays and provides convergence analysis for three kinds of MAS with fixed and switching topologies(13). (14) develops a passivity-based design framework to process the group coordination problem, where both fixed and time-varying communication structures are considered. (15) proposes that all agents reach a consensus if a small fraction of them are controlled by simple feedback control. (16) investigates the robust consensus problems of second-order MAS with diverse input delays and decentralized consensus conditions are obtained for the MAS with symmetric coupling weights based on frequency-domain analysis. (17) investigates the consensus problem for directed MAS with external disturbances and model uncertainties for fixed and switching topologies. (18) studies the average consensus problem for undirected MAS having communication delays and provides sufficient conditions for the existence of average consensus under bounded communication delays. (19) characterizes a distributed algorithm that asymptotically achieved consensus and provides two discontinuous distributed algorithms that achieve max and min consensus, respectively, in finite time. (20)unifies the consensus in MAS and synchronization in complex dynamical network. It's noted that the agent dynamics in most existing works are often restricted to linear and identical ones. The consensus problem of MAS with non-identical agent dynamics is much more complicated than the identical case and few results have been reported to date.

Inspired by these early results, the present paper will focus on the global consensus problems of MAS, and the proposed consensus property is formulated in terms of certain boundedness of state errors. Compared with many existing related results, this paper make two significant advances. One is that we generalize the related results for the case of identical agent dynamics to the case of non-identical agent dynamics. The other is we design the nonlinear feedback controllers to guarantee the MAS achieve exact consensus.

The rest of this paper is organized as follows. A continuous-time MAS model with non-identical agent dynamics is presented and some preliminaries are introduced in Section 2. The main results including bounded consensus and exact consensus criterion are derived in Section 3. In Section 4, a numerical simulation example is given to verify the effectiveness of the proposed results, followed by conclusions in Section 5.

2. PRELIMINARIES

2.1 Problem description

Let $G = (\mathcal{V}, \mathcal{A})$ be a graph of order N consists of a set of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and a set of edges $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$. An edge (v_j, v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{A}\}$.

We consider a MAS consists of N non-identical agents with communication delay:

$$\dot{x}_i(t) = B_i x_i(t) + c \sum_{j \in \mathcal{N}_i}^{N} a_{ij} \Gamma x_j(t) + u_i, \quad i = 1, 2, \cdots, N(1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $B_i \in R^{n \times n}$ are constant matrices, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n \times n}$ is the inner coupling matrix, and if $\gamma_{ij} \neq 0$, then it means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N \times N}$, which is symmetric and irreducible, representing the communication topology relation of the MAS, is defined by $a_{ij} = a_{ji} = 1 (v_j \in \mathcal{N}_i)$, $a_{ij} = 0 (v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j \neq i} a_{ij}$. τ is a constant coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The desired moving trajectory is chosen as

$$\dot{s}(t) = \frac{1}{N} \sum_{k=1}^{N} B_k s(t). \tag{2}$$

We'll now discuss the problem of global consensus for the MAS (1). The consensus problem formulation in the present paper is quite different from many others, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. The consensus problem here will be depicted instead via certain boundedness of $x_i(t) - x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. At the same time, we will design controllers for the MAS (1) to guarantee it achieve exact consensus. To address these cases we will focus on making the states of all agents converge to a bounded set or an equilibrium point.

2.2 Mathematical preliminaries

Before stating the main results of this paper, the following mathematical preliminaries are necessary.

Definition 1((21)): The solution $x_i(t, t_0, \psi_i(t))$ of the MAS (1) is said to be uniformly ultimately bounded with respect to the bound ε if for each $\delta > 0$ there exists $T = T(\varepsilon, \delta) > 0$ independent of t_0 such that $\|x_i(t, t_0, \psi_i(t))\| \leq \varepsilon$ for all $t \geq t_0 + T$ when $\|x_i(t_0)\| < \delta$, where $\psi_i(t)$ is the initial value given as $x_i(t) = \psi_i(t)$ for $t \in [t_0 - \tau, t_0]$, $i = 1, 2, \cdots, N$.

Lemma 1((22)): Assuming that the graph $G = (\mathcal{V}, \mathcal{A})$ is a strongly connected graph, then there exists a unitary matrix $\Phi = (\varphi_{ij})_{N \times N} = (\Phi_1, \Phi_2, \cdots, \Phi_N)$, such that the adjacency matrix A satisfies

$$\Phi^T A \Phi = \Lambda = diag\{\lambda_1, \lambda_2, \cdots, \lambda_N\}, \tag{3}$$

where Φ_i is the i-th column of Φ with $\Phi_1 = (\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}})^T$ and $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$ are the eigenvalues of A.

Lemma 2 ((22)): Let $g(t)$ be a non-negative bounded function defined on R^+ and

$$\Omega = \{x(t) \in R^n | \|x(t)\| \leq \overline{lim}_{t \to \infty} g(t)\}. \tag{4}$$

Suppose there exist a strictly positive definite matrix $P(t) \in \mathcal{PC}^1_{n \times n}$ and a constant $\delta > 0$ such that the derivative of $V(x(t), t) = x^T(t) P(t) x(t)$ along the trajectory of the system

$$\dot{x}(t) = f(x(t), t), \quad x(t) \in R^n, t \in [0, \infty) \tag{5}$$

satisfies

$$\dot{V} \leq -\delta \|x(t)\|^2 \;\; \texttt{if} \;\; \|x(t)\| \geq g(t). \tag{6}$$

For any $t > 0$, let

$$Q_t = \{x(t) | V(x(t), t) \leq \sup_{y(s) \in \Omega, s \geq 0} \{V(y(s), s)\}\} \tag{7}$$

and

$$c = \overline{lim}_{t \to \infty} (max\{\|x(t)\| | x(t) \in Q_t\}). \tag{8}$$

Then, $x(t)$ converges to the set

$$M = \{x(t) | \|x(t)\| \leq c\}. \tag{9}$$

We now introduce some notations and definitions.

Let $PC^1_{n \times n}$ be the linear space of the uniformly bounded continuous real matrix-valued functions defined on $[0, \infty)$. For any $P \in PC_{n \times n}$ the norm of P is defined by $\|P\| = \max_{0 \leq t < \infty} \{\|P(t)\|\}$.

3. MAIN RESULTS

By defining the consensus error vector as

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \cdots, N. \tag{10}$$

Obviously, $\frac{c}{N} \sum_{k=1}^{N} \sum_{j \in \mathcal{N}_i}^{N} a_{kj} \Gamma x_j(t - \tau) = 0$, from (1) and (2), the error dynamics are found to be

$$\dot{e}_i(t) = B_i x_i(t) - \frac{1}{N} \sum_{k=1}^{N} B_k x_k(t) + c \sum_{j \in \mathcal{N}_i}^{N} a_{ij} \Gamma x_j(t) \tag{11}$$

for $i = 1, 2, \cdots, N$. Assuming that the trajectories of all agents are bounded, it follows that their difference is also bounded. Then, the following inequalities can be defined

$$\|B_i x_i(t) - \frac{1}{N} \sum_{k=1}^{N} B_k x_k(t)\| \leq \alpha_i \tag{12}$$

for $i = 1, 2, \cdots, N$, where α_i are nonnegative constants.

In order to achieve consensus, the local controllers u_i have to be designed such that $e_i(t)$ becomes asymptotically stable about its zero fixed point. Here, we will consider the problems of consensus analysis, controlled bounded consensus and controlled exact consensus respectively.

3.1 Bounded consensus analysis

We firstly consider the problem of consensus analysis of the MAS (1) and the main result of this contribution can be stated as follows:

Theorem 1. If $c > \frac{-1}{\lambda_i}(\lambda_i \neq 0)$ and $0 < \delta < 1$, then the MAS(1) will achieve bounded consensus and the consensus set is depicted as follows

$$M = \{e(t) | \|e(t)\| \leq \frac{\sum_{j=1}^{N} \alpha_j \overline{lim}_{t\to\infty} \|w_{ji}(t)\|}{1-\delta}\}, \quad (13)$$

namely, $e_i(t) = x_i(t) - s(t) \to \Omega$ as $t \to \infty$, where λ_i are the eigenvalues of A.

Proof. Discarding the controllers and rewriting the error system (11) in vector form one gets:

$$\dot{\bar{e}}(t) = \bar{B}(t) + c\Gamma\bar{e}(t)A^T, \quad (14)$$

where $\bar{e}(t) = [e_1(t), e_2(t), \cdots, e_N(t)] \in R^{n\times N}$, $\bar{B}(t) = [\hat{B}_1(t), \hat{B}_2(t), \cdots, \hat{B}_N(t)] \in R^{n\times N}$, $\hat{B}_i(t) = B_i x_i(t) - \frac{1}{N}\sum_{k=1}^{N} B_k x_k(t)$, $i = 1, 2, \cdots, N$.

Given that the connectivity matrix satisfies Lemma 1, there are two matrices, $\Omega = (w_1, w_2, \cdots, w_N) \in R^{N\times N}$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_N) \in R^{N\times N}$ such that:

$$A = \Omega^T \Lambda \Omega, \quad (15)$$

where λ_i and w_i are the $i-$th eigenvalue and associated eigenvector of A respectively. With $\Omega^T\Omega = I_N$, the $N-$dimensional identity matrix.

Using a change of variables $\bar{\eta}(t) = \bar{e}(t)\Omega^T$, the error dynamics become

$$\dot{\bar{\eta}}(t) = \bar{B}(t)\Omega^T + c\Gamma\bar{\eta}(t)\Lambda,$$

where $\bar{\eta}(t) = (\eta_1(t), \eta_2(t), \cdots, \eta_N(t))$, with $\eta_i(t) = \bar{e}(t)w_i^* \in R^n$ and $w_i^* = [w_{1i}, w_{2i}, \cdots, w_{Ni}]^T \in R^{N\times 1}$, or equivalently,

$$\dot{\eta}_i(t) = \bar{B}(t)w_i^* + c\Gamma\eta_i(t)\lambda_i, \quad i = 1, 2, \cdots, N. \quad (16)$$

The stability of the error dynamics (14) around the zero fixed point can be determine using the Lyapunov candidate function:

$$V = \frac{1}{2}\sum_{i=1}^{N} \eta_i^T(t)\eta_i(t). \quad (17)$$

The time derivative of V along the trajectories of the error dynamics in (16) is given by

$$\dot{V} = \sum_{i=1}^{N} (\eta_i^T(t)\bar{B}(t)w_i^*(t) + \eta_i^T(t)c\lambda_i\Gamma\eta_i(t)).$$

Considering the bounds of each term of $\dot{V}$. From (12) one has the bound of the first term:

$$\|\bar{B}(t)w_i^*\| \leq \|\sum_{j=1}^{N} B_j(t)w_{ji}\| \leq \|\sum_{j=1}^{N} \alpha_j w_{ji}\|.$$

The second term is always semi-negative because $c > 0$, $\lambda_i \leq 0$ and it can be expressed as

$$c\lambda_i \|\eta_i^T(t)\|\|\Gamma\|\|\eta_i(t)\| \leq -\|\eta_i^T(t)\|\|\eta_i(t)\|$$

if we take $c > \frac{-1}{\lambda_i}(\lambda_i \neq 0)$.

From the above results the time derivative V is bounded by

$$\dot{V} \leq \sum_{i=1}^{N} (\|\eta_i^T(t)\| \sum_{j=1}^{N} \alpha_j \|w_{ji}\| - \|\eta_i^T(t)\|\|\eta_i(t)\|).$$

If we take $\frac{\sum_{j=1}^{N} \alpha_j \|w_{ji}\|}{\|\eta_i(t)\|} < \delta < 1$, then we have

$$\dot{V} \leq (\delta - 1)\|\eta_i(t)\|^2.$$

Applying Lemma 2 completes the proof. In consequence, the MAS (1) achieves bounded consensus.

Now we will consider the controlled consensus problem of MAS (1), and bounded consensus criterion and exact consensus criterion are given as follows respectively.

3.2 Bounded consensus via linear negative feedback

Theorem 2. If the local controllers u_i are constructed as

$$u_i = -ck\Gamma e_i(t),$$

$k > \lambda_i + \frac{1}{c}$ and $0 < \delta < 1$, then the MAS(1) will achieve bounded consensus and the consensus set is depicted as follows

$$M = \{e(t) | \|e(t)\| \leq \frac{\sum_{j=1}^{N} \alpha_j \overline{lim}_{t\to\infty} \|w_{ji}(t)\|}{1-\delta}\}, \quad (18)$$

namely, $e_i(t) = x_i(t) - s(t) \to \Omega$ as $t \to \infty$, where λ_i are the eigenvalues of A.

Proof. The proof is very similar to the Theorem 1, so is omitted here.

3.3 Exact consensus via nonlinear feedback

The designed controllers in the Theorem 2 make the MAS (1) achieve bounded consensus instead of exact consensus, the following theorem will consider how to design controllers to guarantee the MAS achieve exact consensus.

Theorem 3. The MAS(1) will achieve exact consensus if the local controllers u_i are constructed as

$$u_i = -ck\Gamma e_i(t) - \rho sgn(e_i(t)) \quad (19)$$

for $i = 1, 2, \cdots, N$, where $sgn(e_i(t)) = [sgn(e_{i1}(t)), sgn(e_{i2}(t)), \cdots, sgn(e_{in}(t))]^T$, with $sgn(.)$ are signum function, and furthermore, the controller gains are designed to satisfy the bounds

$$k > \max\{\lambda_i + \frac{1}{c}, 0\}$$
$$\rho > \frac{\sum_{j=1}^{N} \alpha_j \|w_{ji(t)}\|}{\sum_{j=1}^{N} \|w_{ji(t)}\|}$$

for any i.

Proof. Rewriting the error system (11) in vector form one gets:

$$\dot{\bar{e}}(t) = \bar{B}(t) + c\Gamma\bar{e}(t)(A^T - K) - \rho sgn(\bar{e}_i(t)), \quad (20)$$

where $\bar{e}(t) = [e_1(t), e_2(t), \cdots, e_N(t)] \in R^{n\times N}$, $\bar{B}(t) = [\hat{B}_1(t), \hat{B}_2(t), \cdots, \hat{B}_N(t)] \in R^{n\times N}$, $\hat{B}_i(t) = B_i x_i(t) - \frac{1}{N}\sum_{k=1}^{N} B_k x_k(t), i = 1, 2, \cdots, N, K = diag\{k, k, \cdots, k\} \in R^{N\times N}$ and $sgn(\bar{e}(t)) = [sgn(e_1(t)), sgn(e_2(t)), \cdots, sgn(e_N(t))] \in R^{n\times N}$.

Given that the connectivity matrix satisfies Lemma 1, there are two matrices, $\Omega = (w_1, w_2, \cdots, w_N) \in R^{N\times N}$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_N) \in R^{N\times N}$ such that:

$$A = \Omega^T \Lambda \Omega, \quad (21)$$

where λ_i and w_i are the $i-$th eigenvalue and associated eigenvector of A respectively. With $\Omega^T\Omega = I_N$, the $N-$dimensional identity matrix.

Using a change of variables $\bar{\eta}(t) = \bar{e}(t)\Omega^T$, the error dynamics become

$$\dot{\bar{\eta}}(t) = [\bar{B}(t) - \rho sgn(\bar{\eta}(t)\Omega)]\Omega^T + c\Gamma\bar{\eta}(t)(\Lambda - K),$$

where $\bar{\eta}(t) = (\eta_1(t), \eta_2(t), \cdots, \eta_N(t))$, with $\eta_i(t) = \bar{e}(t)w_i^* \in R^n$ and $w_i^* = [w_{1i}, w_{2i}, \cdots, w_{Ni}]^T \in R^{N\times 1}$, or equivalently,

$$\dot{\bar{\eta}}_i(t) = (\bar{B}(t) - \rho sgn(\bar{\eta}(t)\Omega))w_i^* + c\Gamma\eta_i(t)(\lambda_i - k) \quad (22)$$

for $i = 1, 2, \cdots, N$.

The stability of the error dynamics (20) around the zero fixed point can be determine using the Lyapunov candidate function:

$$V = \frac{1}{2}\sum_{i=1}^{N} \eta_i^T(t)\eta_i(t). \quad (23)$$

The time derivative of V along the trajectories of the error dynamics in (16) is given by

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N}(\eta_i^T(t)\bar{B}(t)w_i^*(t) - \bar{\eta}_i^T(t)\rho sgn(\bar{\eta}(t)\Omega)w_i^*(t) \\ &\quad + \eta_i^T(t)c(\lambda_i - k)\Gamma\eta_i(t)) \\ &\leq \sum_{i=1}^{N}(\|\eta_i^T(t)\|\|\bar{B}(t)w_i^*(t)\| - \rho\|\bar{\eta}_i^T(t)\|\|sgn(\bar{\eta}(t)\Omega)w_i^*(t)\| \\ &\quad + c(\lambda_i - k)\|\eta_i^T(t)\|\Gamma\|\eta_i(t)\|).\end{aligned}$$

Considering the bounds of each term of $\dot{V}$. From (12) one has the bound of the first term:

$$\|\bar{B}(t)w_i^*(t)\| \leq \|\sum_{j=1}^{N} B_j(t)w_{ji}(t)\| \leq \|\sum_{j=1}^{N}\alpha_j w_{ji}(t)\|.$$

The bound for the second term is given by:

$$\begin{aligned}\|sgn(\bar{\eta}(t)\Omega)w_i^*(t)\| &\leq \sum_{j=1}^{N}\|sgn(\bar{\eta}(t)w_j(t))\|\|w_{ji}(t)\| \\ &\leq \sum_{j=1}^{N}\|w_{ji}(t)\|.\end{aligned}$$

The third term is quadratic and will be negative if the coefficient is negative ($c(\lambda_i - k) < 0$) for any i. The bound on the third term can be expressed as

$$c(\lambda_i - k)\|\eta_i^T(t)\|\Gamma\|\eta_i(t)\| \leq -\|\eta_i^T(t)\|\|\eta_i(t)\|.$$

From the above results the time derivative V is bounded by

$$\begin{aligned}\dot{V} &\leq \sum_{i=1}^{N}(\|\eta_i^T(t)\|(\sum_{j=1}^{N}\alpha_j\|w_{ji}(t)\| - \rho\sum_{j=1}^{N}\|w_{ji}(t)\|) \\ &\quad - \|\eta_i^T(t)\|\|\eta_i(t)\|).\end{aligned}$$

For V to be negative, the discontinuous gain must satisfy $\rho > \frac{\sum_{j=1}^{N}\alpha_j\|w_{ji}(t)\|}{\|w_{ji}(t)\|}$ for any i. Then the error dynamics (22) are globally uniformly asymptotically stable about the zero fixed point ($\bar{\eta}(t) = 0$), which implies that the MAS (1) under the controllers (19) achieve consensus.

Remark 1. The above bounded consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical nodes to the cases of non-identical nodes. Because of the complexity of the consensus problems for non-identical nodes, we only obtain here sufficient conditions instead of sufficient and necessary condition. However, the conditions obtained here are easy to verify and we can easily construct appropriate numerical simulation example to verify the effectiveness of the proposed results. Comparing the above three theorems, it can be seen that the linear negative feedback can't guarantee the MAS achieve exact consensus, at the same time, the boundary of the convergence set can be evaluated respectively.

4. EXAMPLE

In this section, we will construct an example to demonstrate the proposed results above. The problem is to guarantee 11 agents to follow desired curves in a 2-dimensional system of coordinate.

The agent dynamics can be chosen as follows

$$\dot{x}_i(t) = B_i x_i(t), \quad i = 1, 2, \cdots, 11, \quad (24)$$

where

$$B_i = \begin{pmatrix} -10 + 0.1\times(i-1) & 10 - 0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15 - 0.1\times(i-1) & 0 \end{pmatrix}.$$

The communication coupling matrix A and the inner coupling matrix are

$$A=\begin{pmatrix} -8 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -8 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -6 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & -7 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -5 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -10 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -6 \end{pmatrix}$$

and $\Gamma = diag\{1, 1, 1\}$respectively.

Given the initial values of 11 agents as $(10\ 5\ \ -10)^T$, $(12\ 6\ \ -12)^T$, $(14\ 7\ \ -14)^T$, $(16\ 8\ \ -16)^T$, $(18\ 9\ \ -18)^T$, $(20\ 10\ \ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$ and $(-10\ 15\ 10)^T$ respectively. We may verify the conditions of Theorem 3 readily, then the consensus of the MAS is achieved. Simulation results are depicted in Fig.1 to Fig.5 for $c = 1$, $k = 0.5$.

The simulation curves in Fig.1-Fig.6 show that the dynamics of the MAS in different time scale with and without control respectively. The average state trajectory $s(t)$ is chosen as the desired moving trajectory and is depicted in Fig.3 and Fig.6 respectively. These simulation curves show that all agents eventually move with the desired state trajectory in the sense of boundedness on the condition that there is not control in the MAS or by using linear negative feedback. The exact consensus can be guaranteed by means of nonlinear controllers.

5. CONCLUSIONS

In this paper, we've investigated the consensus problems of MAS with different node dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the MAS is achieved based on a series of transformations and Lyapunov stability theorem. The methods we present here have several distinct features. Firstly, they are very simple in form, but are more effective to resolve the consensus problem with nonidentical node dynamics. Secondly, the proposed nonlinear controllers can guarantee the MAS achieve exact consensus instead of in terms of boundedness. It should be noted that the conditions are still restrictive and there is no timedelay in the communication topology, further work regarding this topic we'll focus on these problems. An obvious limitation of the proposed method is the fact the requires the same number of controllers than nodes, in a work to be reported elsewhere this controlled consensus design is combined with a pinning control strategy, providing a reduction on the number of nodes where controlled action in taken. Yet another aspect of interest to considered as future work is determining conditions for the existence of an appropriate coordinate transformation to translate the nonlinear agent dynamics to linear ones such that this result is applicable for a more general class of oscillators.

REFERENCES

[1] Sandro Zampieri. Trends in networked control systems[C], *Proceedings of 17th World Congress of IFAC*, Seoul, Korea, July 6-11, 2008, pp. 2886 – 2894.

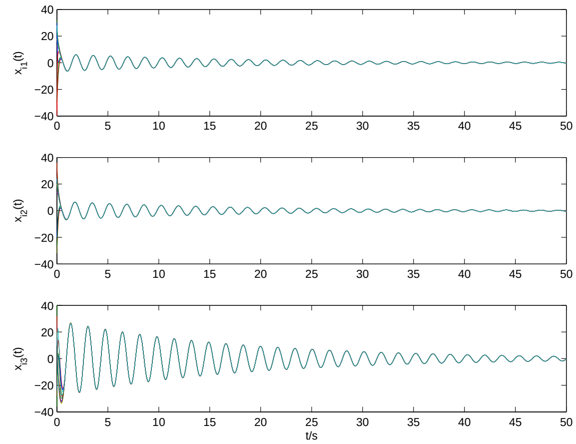

Fig.1. The dynamics of all agents without control.

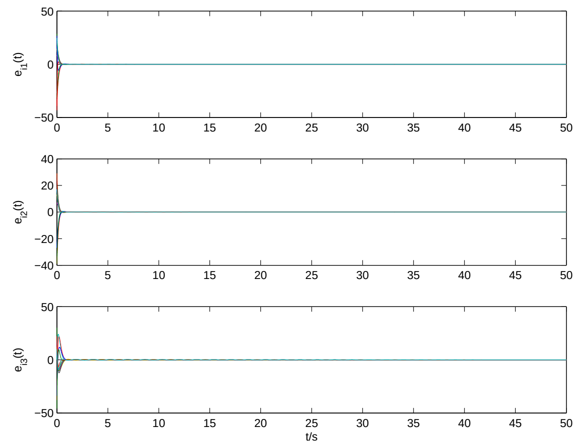

Fig.2. The consensus error dynamics of each agent without control.

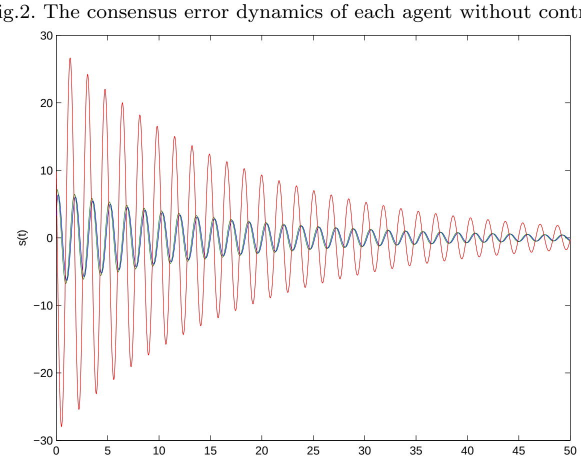

Fig.3. The average state trajectory $s(t)$ without control.

[2] J.P. Desai, J.P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust[J]. *IEEE Int. Trans. on Robotics and Automation*, 2001, 17(6): 905 – 908.

[3] W. Ren, R.W. Beard. Formation feedback control for multiple spacecraft via virtual structures[J]. *IEE Proc. Control Theory and Applications*, 2004, 151(3): 357 – 368.

[4] H. Yamaguchi, T. Arai and G. Beni. A distributed control scheme for multiple robotic vehicles to make

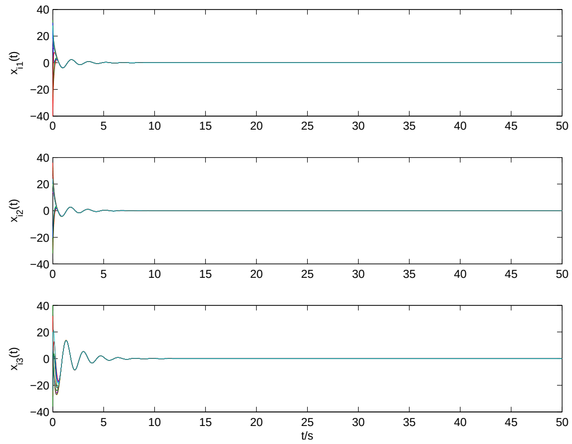

Fig.4. The dynamics of all agents under control.

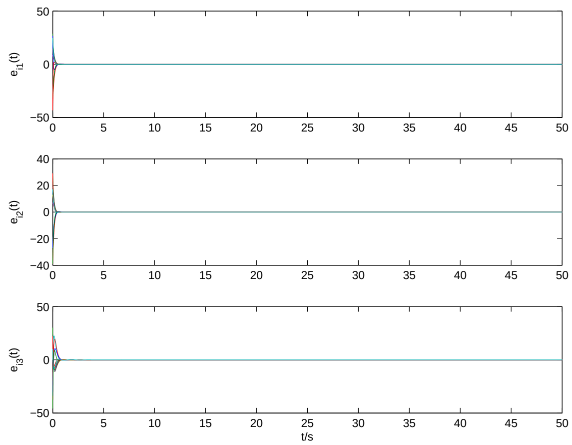

Fig.5. The consensus error dynamics of each agen under control.

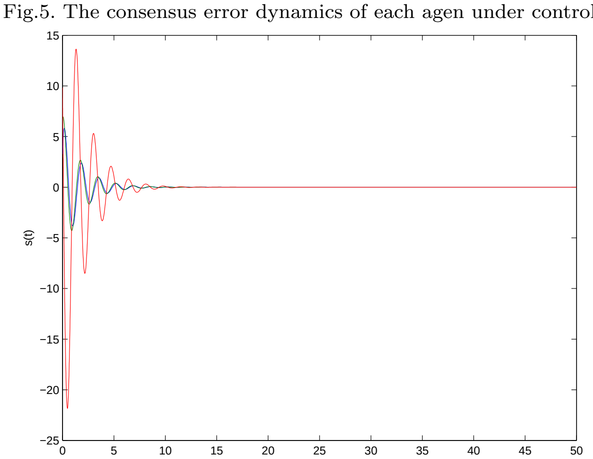

Fig.6. The average state trajectory $s(t)$ under control.

group formations[J]. *Robitics and Autonomous Systems*, 2001, 36: 125-147.

[5] Z.P. Wu, Z.H. Guan, T. Li. Group motion control of multi-agent systems based on complex network[C], *Proceedings of the 26th Chinese Control Conference*, Hunan, China, July 26-31, 2007, pp: 63 – 67.

[6] M. Porfiri, D.G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach[J]. *Automatica*, 2007, 43: 1318 – 1328.

[7] J. Cortés. Global formation-shape stabilization of relative sensing networks[C]. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 1460 – 1465.

[8] Y. Liu, Y.M. Jia, J.P. Du and S.Y. Yuan. Dynamic output feedback control for consensus of multi-agent systems: an H_∞ approach[C]. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 4470 – 4475.

[9] Y.G. Hong, J.P. Hu and L.X. Gao. Tracking control for multi-agent consensus with an active leader and variable topology[J]. *Automatica*, 2006, 42: 1177 – 1182.

[10] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays[J]. *IEEE Transactions on Automatic Control*, 2008, 53(8): 1804 – 1816.

[11] T. Li and J.F. Zhang. Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash equilibria[J]. *Automatica*, 2008, 44: 713 – 725.

[12] E.S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information[J]. *Automatica*, 2008, 44: 2766 – 2777.

[13] R. Olfati-Saber and R.M. Murray. Consensus problmes in networks with switching topology and time-delays[J]. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1520 – 1533.

[14] M. Arack. Passivity as a design tool for group coordination[J]. *IEEE Transactions on Automatic Control*, 2007, 52(8): 1380 – 1390.

[15] F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu and Z.Z. Yuan. Reaching a consensus via pinning control[J]. *Automatica*, 2009, 45: 1215 – 1220.

[16] Y.P. Tian and C.L Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations[J]. *Automatica*, 2009, 45: 1347 – 1353.

[17] P. Lin, Y.M. Jia and L. Li. Distributed robust H_∞ Consensus control in directed networks of agents with time-delay[J]. *Systems and Control Letters*, 2008, 57(8): 643 – 653.

[18] P.A. Bliman and G.F. Trecate. Average consensus problems in networks of agents with delayed communications[J]. *Automatica*, 2008, 44: 1985 – 1995.

[19] J. Cortes. Distributed algorithms for reaching consensus on general functions[J]. *Automatica*, 2008, 44(3): 726 – 737.

[20] Z.K. Li, Z.S. Duan, G.R. Chen and L. Huang. Consensus of multi-agent systems and synchronization of complex networks: a unified viewpoint[J]. *IEEE Transactions on Circuits and Systems-I*, 2009 (to appear).

[21] C.C. Hua, X.P. Guan and P. Shi. Decentralized robust model reference adaptive cotnrol for interconnected time-delay systems[J]. *J.Comput.Appl.Math.*, 2006, 193: 383 – 396.

[22] D. J.Hill and J. Zhao. Global synchronization of complex dynamical networks with non-identical nodes[C], *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 9-11, 2008, 817 – 822.

:

Global Consensus Analysis of Networked Multi-Agent Systems with Heterogeneous Dynamics and Time-Varying Communication Delay

Wei-Song ZHONG* and Guo-Ping LIU*,**, *Fellow, IEEE*

**The Faculty of Advanced Technology,*
University of Glamorgan, CF37 1DL, Pontypridd, U.K.
***CTGT Center,*
Harbin Institute of Technology, 150001, Harbin, China.

Abstract

This paper investigates the global bounded consensus problem of the networked Multi-Agent Systems exhibiting nonlinear, non-identical agent dynamics and time-varying delay in the communication channels. Based on the Lypunov-Krasovskii functional method, globally bounded consensus conditions for both delay-independent and delay-dependent are derived respectively. The proposed consensus criteria ensures that all agents eventually move along the desired trajectories involved with the average agent dynamics in the sense of boundedness. The proposed consensus criteria generalizes the case of networked Multi-Agent Systems with identical agent dynamics to the case of non-identical agent dynamics, and many existing results related to this area can be viewed as special cases of the present results. Finally, a numerical simulation example is constructed to demonstrate the effectiveness of the proposed theoretical results.

Keywords: Networked Multi-Agent Systems, Consensus, Time-Varying Delay

1. INTRODUCTION

Networked Multi-Agent Systems (MAS) analysis involves the study of how the network architectures and interactions between network components influence global control goals. The consensus problem in MAS requires an agreement to be reached that depends on the states of all agents. The topic has been studied across many fields of science and engineering [1]-[12]. Olfati-Saber introduced two consensus criteria for networks with and without time-delays and provided convergence analysis for three kinds of MAS with fixed and switching topologies [13]. A passivity-based design framework is developed to process the group coordination problem, where both fixed and time-varying communication structures are considered in [14]. All agents reach a consensus if a small fraction of them are controlled by simple feedback control is proposed in [15]. The robust consensus problems of second-order MAS with diverse input delays are investigated and decentralized consensus conditions are obtained for the MAS with symmetric coupling weights based on frequency-domain analysis in [16]. The consensus problem for directed MAS with external disturbances and model uncertainties for fixed and switching topologies are discussed in [17]. The average consensus problem for undirected MAS having communication delays is studied and sufficient conditions are provided for the existence of average consensus under bounded communication delays in [18]. A distributed algorithm that asymptotically achieved consensus is characterized and two discontinuous distributed algorithms that achieve max and min consensus are provided respectively in [19]. It is noted that the agent dynamics in most existing works are often restricted to linear and identical ones. Obviously, in practice, this is not always the case. The consensus problem of MAS with non-identical agent dynamics is much more complicated than the identical case and few results have been reported to date.

The present paper will focus on the global consensus problems of the MAS consisting of heterogenous agents. The behavior of the MAS with non-identical agent dynamics is much more complicated than the identical case. Usually, no common equilibrium for all agents exists even if each agent has an equilibrium, neither does a consensus manifold exist in the classical sense. Consensus of a MAS with identical agents is usually described in terms of (asymptotically) identical dynamical evolution of state variables of every agent in the MAS, which is easy to understand. However, this collective behavior, called exact consensus no longer exists in the MAS with non-identical agents due to the difference between the dynamics of the agents. Furthermore, the MAS with non-identical agent dynamics cannot be decoupled into a number of lower dimensional systems exactly like the identical-agent case. Yet, a MAS with non-identical agents may still exhibit some kinds of consensus behaviors which are far from being fully understood. Certain reasonable and satisfactory boundedness of state motions errors between different agents can be taken as useful consensus properties. Compared with many existing results, this paper makes several significant advances. Firstly, the related results for the case of identical agent dynamics has been generalized to the case of non-identical agent dynamics and the proposed results cover many existing criteria related to this area. Secondly, the time-varying delay has been introduced in the communication channels and global consensus criteria are given based on solving a number of lower dimensional matrix inequalities and scalar inequalities. Finally, globally bounded consensus conditions for both delay-independent and delay-dependent are derived based on the Lypunov-Krasovskii functional method . The rest of this paper is organized as follows. A continuous-time MAS model with non-identical agent dynamics and communication time-varying delay and the objective are presented in Section II. The main results including delay-independent and delay-dependent bounded consensus criterion are derived in Section III and IV. In Section V, a numerical simulation example is given to verify the effectiveness of the proposed results, followed by conclusions in Section VI.

2. PROBLEM DESCRIPTION

Let $G = (\mathcal{V}, \mathcal{A})$ be a graph of order N consisting of a set of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and a set of edges $\mathcal{A} \subseteq \mathcal{V}\times\mathcal{V}$. An edge (v_j, v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{A}\}$.

We consider a MAS consisting of N non-identical agents with time-varying communication delay:

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma x_j(t-\tau(t)) \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i(t)) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix, and where $\gamma_{ij} \neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N\times N}$(which is symmetric and irreducible) represents the communication topology relation of the MAS, and is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j\neq i} a_{ij}$. $\tau(t)$ is a time-varying coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field

$$\bar{f}(x(t)) = \frac{1}{N}\sum_{k=1}^{N} f_k(x(t)). \tag{2}$$

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t). \tag{3}$$

We now discuss the problem of global consensus for the MAS (1). The consensus problem formulation in the present paper is quite different from many others, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. The consensus problem here will be depicted instead via certain boundedness of $x_i(t)-x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. This better reflects reality since it is impossible for MAS (1) to achieve exact consensus. To address this case we will focus on making the states of all agents converge to a bounded set.

Define the error vector

$$e_i(t) = x_i(t) - s(t), \quad i = 1, 2, \cdots, N. \tag{4}$$

Obviously, $\sum_{i=i}^{N} e_i = 0$ and $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i} a_{kj}\Gamma x_j(t-\tau(t)) = 0$, then the MAS (1) can be rewritten in terms of e_i as

$$\dot{e}_i(t) = f_i(s(t) + e_i(t)) - \frac{1}{N}\sum_{k=1}^{N} f_k(s(t) + e_k(t)) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau(t)). \tag{5}$$

Let $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$ and then the error system (5) can be written further as follows by means of the Newton-Leibniz formula.

$$\dot{e}(t) = I_N\otimes D\bar{f}(s)e(t) + cA\otimes\Gamma e(t-\tau(t)) + I(t)e(t) - \frac{1}{N}H(t)e(t) + F(t), \tag{6}$$

where $I(t) = diag\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau, \cdots, \int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\}$, $\bar{\Sigma}(t) = I_N\otimes D\bar{f}(s)+D$, $H^T(t) = (H_1^T(t), \cdots, H_N^T(t))$, $H_i(t) = (\int_0^1 Df_1(s+\tau e_1)d\tau, \cdots, \int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t) = (f_1^T(s) - \bar{f}^T(s), \cdots, f_N^T(s) - \bar{f}^T(s))$.

Repeating the similar simplification procedure proposed in [20], [21], the above error system can be transformed into the following $N-1$ relatively simple system after a series of transformation.

$$\begin{aligned}\dot{\omega}_i(t) =& D\bar{f}(s(t)\omega_i(t) + c\lambda_i\Gamma\omega_i(t-\tau(t)) \\ &+ (\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)\omega(t) + (\Phi_i^T\otimes I_n)F(t),\end{aligned} \tag{7}$$

where $i = 2, 3, \cdots, N$.

So far, the consensus problem of the MAS (1) has been transferred to the stability problem of the $N-1$ of $n-$dimensional systems.

3. DELAY-INDEPENDENT BOUNDED CONSENSUS CRITERIA

A delay-independent bounded consensus criteria for the MAS (1) is given as follows.

Theorem 3.1 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$\begin{aligned}a\|x(t)\|^2 \leq& x^T(t)P_i(t)x(t) + \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \\ &\leq b\|x(t)\|^2, \ \forall t\in R^+, \ x\in R^n, i = 2, 3, \cdots, N,\end{aligned} \tag{8}$$

$$\begin{aligned}&\dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + Q_i \\ &+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \zeta I \leq 0, \ i = 1, 2, \cdots, N,\end{aligned} \tag{9}$$

$$\|I(t)\| \leq \gamma,\ i = 1, 2, \cdots, N. \tag{10}$$

Then the system (6) converges to the set

$$M = \{e(t) | \|e(t)\| \leq \frac{2b}{a} \frac{\beta \overline{lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\} \tag{11}$$

for any fixed time delay $\tau > 0$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, $\beta = (\sum_{i=2}^{N} \|P_i(t)\|^2)^{\frac{1}{2}}$, $\zeta > 2\gamma\beta$ and $\mu(t) = \|F(t)\|$ is bounded. Furthermore, the MAS (1) achieves bounded consensus for any fixed time delay $\tau(t) > 0$, $0 \leq \dot{\tau}(t) \leq 1$.

Proof. Choose the following Lyapunov-Krasovskii functional as

$$V(w_i(t)) = \sum_{i=2}^{N} V_i(w_i(t)), \tag{12}$$

$$V_i(w_i(t)) = w_i^T(t)P_i(t)w_i(t) + \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha. \tag{13}$$

Differentiating (13) along the trajectory of (7) gives

$$\begin{aligned}
&\dot{V}_i(w_i(t))\\
&= w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s) + D\bar{f}(s)^T P_i(t) + Q_i)w_i(t)\\
&\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t)\\
&\quad + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t)\\
&\quad + 2w_i^T(t)(c\lambda_i P_i(t)\Gamma)w_i(t-\tau(t))\\
&\quad - (1-\dot{\tau}(t))w_i^T(t-\tau(t))Q_i w_i(t-\tau(t)).
\end{aligned} \tag{14}$$

Applying the Young Inequality to the equality (14) resulting in

$$\begin{aligned}
\dot{V}_i(w_i(t)) \leq & w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s) + D\bar{f}(s)^T P_i(t)\\
& + Q_i + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i(t)\\
& + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w(t)\\
& + 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t).
\end{aligned} \tag{15}$$

Condition (9) implies that the first term on the right hand side of (15) satisfies

$$\begin{aligned}
& w_i^T(t)(\dot{P}_i(t) + P_i(t)D\bar{f}(s) + D\bar{f}(s)^T P_i(t) + Q_i\\
&\quad + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i(t) \leq -\zeta\|w_i(t)\|^2.
\end{aligned} \tag{16}$$

Applying condition (10) we know the second term on the right hand side of (15) satisfies

$$\begin{aligned}
& 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t)\\
&\qquad \leq 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|.
\end{aligned} \tag{17}$$

The third term on the right hand side of (15) satisfies

$$2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i(t)\|. \tag{18}$$

Since $V(w(t)) = \sum_{i=2}^{N} V_i(w_i(t))$, we have

$$\begin{aligned}
&\dot{V}(w(t)) = \sum_{i=2}^{N} \dot{V}_i(w_i(t))\\
&\leq \sum_{i=2}^{N}(-\zeta\|w_i(t)\|^2) + 2\gamma\|P_i(t)\|\|w_i(t)\|\|w(t)\|\\
&\quad + 2\mu(t)\|P_i(t)\|\|w_i(t)\|\\
&= -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\| + \mu(t))\sum_{i=2}^{N}\|w_i(t)\|\|P_i(t)\|\\
&\leq -\zeta\|w(t)\|^2 + 2(\gamma\|w(t)\| + \mu(t))\|w(t)\|(\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}}\\
&= \|w(t)\|((2\gamma\beta - \zeta)\|w(t)\| + 2\beta\mu(t)).
\end{aligned} \tag{19}$$

Thus, when

$$\|w(t)\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \tag{20}$$

we have

$$\dot{V}(w(t)) \leq -\delta\|w(t)\|^2. \tag{21}$$

Applying the results in [21] completes the proof.

Remark 3.1 We have an asymptotic consensus criterion in the classical sense when $\overline{lim}_{t\to\infty}\mu(t) = 0$. In particular, we have $\mu(t) \equiv 0$ when all agents have the same dynamics, i.e., $f_i(x_i(t)) = f(x(t))$. In such a case, applying Theorem 3.1 to the linearized network, which is equivalent to taking $\gamma = 0$ in (10), immediately achieves the universal consensus criteria existing in many literatures. Therefore, Theorem 3.1 covers the existing criteria of networks with identical agent dynamics as a special case.

Remark 3.2. The above result is a delay-independent globally consensus criterion and the ultimate convergence bound is evaluated by means of (11). Theorem 3.1 guarantees that all agents move along the desired trajectory described by $s(t)$ in terms of certain boundedness, i.e., the consensus achieved here is just approximate instead of exact, in fact, to achieve exact consensus is impossible for such a case.

4. DELAY-DEPENDENT BOUNDED CONSENSUS CRITERIA

Delay-dependent criterion for the MAS (1) will be discussed in this section.

Theorem 4.1 Suppose that (8) and (10) in Theorem 3.1 are satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Pi_i > 0$, $\Pi_i > 0$, X_i, Y_i and Z_i of appropriate dimensions such that

$$\Xi = \begin{pmatrix} \Xi_1 & \Xi_2 \\ \Xi_2^T & \Xi_3 \end{pmatrix} \tag{22}$$

where $\Xi_1 = \dot{P}_i(t) + P_i(t)D\bar{f}(s(t)) + D\bar{f}(s(t))^T P_i(t) + hX_i + Y_i^T + Y_i + Q_i + hD\bar{f}(s(t))^T Z_i D\bar{f}(s(t))$, $\Xi_2 =$

$c\lambda_i P_i(t)\Gamma - Y_i + hc\lambda_i D\bar{f}(s(t))^T Z_i\Gamma$, $\Xi_3 = \Pi_i^{-1} + \Sigma_i^{-1} - Q_i + hc^2\lambda_i^2\Gamma^T Z_i\Gamma$ and

$$\begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \geq 0, \tag{23}$$

for $i = 2, 3, \cdots, N$, then the MAS (1) will achieve bounded consensus for the time varying delay $0 \leq \tau(t) \leq h < \infty$, $0 < \dot{\tau}(t) \leq 1$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V(w_i(t)) = \sum_{i=2}^{N}\sum_{k=1}^{3} V_k(w_i(t)), \tag{24}$$

where

$$\begin{aligned} V_1(w_i(t)) &= w_i^T(t)P_i(t)w_i(t), \\ V_2(w_i(t)) &= \int_{-\tau(t)}^{0}\int_{t+\beta}^{t} \dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha d\beta, \\ V_3(w_i(t)) &= \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha. \end{aligned}$$

The i-th ($i = 2, 3, \cdots, N$) equation in system (7) can be written as

$$\begin{aligned} \dot{\omega}_i(t) = (D\bar{f}(s(t)) + c\lambda_i\Gamma)\omega_i(t) - c\lambda_i\Gamma\int_{t-\tau(t)}^{t}\dot{w}_i(\alpha)d\alpha \\ + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)\omega(t) + (\Phi_i^T \otimes I_n)F(t), \end{aligned} \tag{25}$$

thus the derivative of $V_1(w_i(t), t)$ satisfies

$$\begin{aligned} \dot{V}_1(w_i(t)) = w_i^T(t)(&\dot{P}_i(t) + P_i(t)(D\bar{f}(s) + c\lambda_i\Gamma) \\ &+ (D\bar{f}(s) + c\lambda_i\Gamma)^T P_i(t))w_i(t) \\ &- 2c\lambda_i w_i^T(t)P_i(t)\Gamma\int_{t-\tau(t)}^{t}\dot{w}_i(\alpha)d\alpha \\ &+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \\ &+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t). \end{aligned} \tag{26}$$

Defining $a(.)$, $b(.)$ and M as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$ and $M = c\lambda_i P_i(t)\Gamma$ for all $\alpha \in [t-\tau(t), t]$ and then applying the results in [23] results in

$$\begin{aligned} \dot{V}_1(w_i(t)) \leq\, & w_i^T(t)[\dot{P}_i(t) + P_i(t)D\bar{f}(s) + D\bar{f}(s)^T P_i(t) \\ &+ hX_i + Y_i^T + Y_i]w_i(t) + \int_{t-\tau(t)}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha \\ &+ 2w_i^T(t)(c\lambda_i P_i(t)\Gamma - Y_i)w_i(t-\tau(t)) \\ &+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w(t) \\ &+ 2w_i^T(t)P_i(t)(\Phi_i^T \otimes I_n)F(t). \end{aligned} \tag{27}$$

Moreover, $\dot{V}_2(w_i(t))$ can be enlarged as

$$\begin{aligned} &\dot{V}_2(w_i(t)) \leq h[D\bar{f}(s)w_i + c\lambda_i\Gamma w_i(t-\tau(t))]^T Z_i[D\bar{f}(s)w_i \\ &+ c\lambda_i\Gamma w_i(t-\tau(t))] + 2h(D\bar{f}(s)w_i)^T Z_i(\Phi_i^T \otimes I_n)I(t) \\ &(\Phi \otimes I_n)w + 2h(D\bar{f}(s)w_i)^T Z_i(\Phi_i^T \otimes I_n)F(t) \\ &+ 2h(c\lambda_i\Gamma w_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ &+ 2h(c\lambda_i\Gamma w_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\ &+ 2h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w)^T Z_i(\Phi_i^T \otimes I_n)F(t) \\ &+ h((\Phi_i^T \otimes I_n)F(t))^T Z_i((\Phi_i^T \otimes I_n)F(t)) \\ &+ h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w)^T Z_i((\Phi_i^T \otimes I_n)I(t) \\ &(\Phi \otimes I_n)w) - \int_{t-\tau(t)}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha. \end{aligned} \tag{28}$$

and

$$\dot{V}_3 = w_i^T Q_i w_i - (1-\dot{\tau}(t))w_i^T(t-\tau(t))Q_i w_i(t-\tau(t)). \tag{29}$$

Applying the Young Inequality, then we have $2h(c\lambda_i\Gamma w_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \leq w_i^T(t-\tau(t))\Pi_i^{-1}w_i(t-\tau(t)) + h^2c^2\lambda_i^2 w^T((\Phi \otimes I_n)^T I(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Pi_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n))w(t)$, and $2h(c\lambda_i\Gamma w_i(t-\tau(t)))^T Z_i(\Phi_i^T \otimes I_n)F(t) \leq w_i^T(t-\tau(t))\Theta_i^{-1}w_i(t-\tau(t)) + h^2c^2\lambda_i^2 F^T(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Theta_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)F(t)$.

Applying these two inequalities and the conditions of the theorem results

$$\begin{aligned} &\dot{V}(w_i(t)) \leq \sum_{i=2}^{N}\begin{pmatrix} w_i(t) \\ w_i(t-\tau(t)) \end{pmatrix}^T \Xi \begin{pmatrix} w_i(t) \\ w_i(t-\tau(t)) \end{pmatrix} \\ &+ 2\mu(t)\beta + (\|w\|(2\gamma\beta + 2h\gamma\|D\bar{f}(s(t))\|\sum_{i=2}^{N}\lambda_{max}(Z_i) \\ &+ 2h\mu(t)\|D\bar{f}(s(t))\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + h\gamma^2\sum_{i=2}^{N}\lambda_{max}(Z_i) \\ &+ h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) \\ &+ h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Theta_i)\lambda_i^2\lambda_{max}^2(Z_i))\|w\| \\ &+ 2h\gamma\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)) + h\mu^2(t)\sum_{i=2}^{N}\lambda_i^2\lambda_{max}(Z_i). \end{aligned} \tag{30}$$

Thus when

$$\|w(t)\| \geq \frac{2\mu(t)\beta + 2h\gamma\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)}{\varpi(t)},$$

we have

$$\dot{V}(w(t)) \leq -\delta\|w(t)\|^2 + h\mu^2(t)\sum_{i=2}^{N}\lambda_{max}(Z_i)\lambda_i^2, \tag{31}$$

where $\varpi(t) = -(2\gamma\beta + 2h\gamma\|D\bar{f}(s(t))\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + 2h\mu(t)\|D\bar{f}(s(t))\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + h\gamma^2\sum_{i=2}^{N}\lambda_{max}(Z_i) + h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) + h^2c^2\mu^2(t)$

$\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Theta_i)\lambda_i^2\lambda_{max}^2(Z_i)) - \delta$ and Ξ is defined in (22). Thus, according to [22] and Lyapunov stability theory, bounded consensus is ultimately achieved. This completes the proof.

Remark 4.1. The above two bounded consensus criteria can be viewed as extensions of the related consensus criteria for the cases of identical nodes to the cases of non-identical nodes. Because of the complexity of the consensus problems for non-identical nodes, only sufficient conditions instead of sufficient and necessary condition can be obtained here. At the same time, the conditions obtained here are somewhat complicated and difficult to verify, but according to certain specific cases, we can construct an appropriate numerical simulation example to verify the effectiveness of the proposed results. Comparing the above two theorems, it can be seen that the boundary of the convergence set and the maximum size of time varying delay can be evaluated respectively.

5. Example

In this section, a numerical simulation example is constructed to demonstrate the effectiveness of the proposed results previously. The objective is to guarantee 11 agents to follow the average dynamics chosen as the desired curves.
The agent dynamics can be chosen as follows

$$\dot{x}_i(t) = B_i x_i(t) + g(x_i(t)), \quad i = 1, 2, \cdots, 11, \qquad (32)$$

where B_1-B_6 can be chosen as

$$\begin{pmatrix} -10 + 0.1 \times (i-1) & 10 - 0.1 \times (i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15 - 0.1 \times (i-1) & 0 \end{pmatrix}$$

and B_7-B_{11} can be chosen as

$$\begin{pmatrix} -10 - 0.1 \times (i-6) & 10 + 0.1 \times (i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15 + 0.1 \times (i-6) & 0 \end{pmatrix}$$

and

$$g(x_i(t)) = (-9.5 sin(\frac{\pi x_{i1}(t)}{3.2} + \pi)\ 0\ 0)^T, \quad i = 1, 2, \cdots, 11.$$

The communication coupling matrix A and the inner coupling matrix Γ are chosen as

$$A = \begin{pmatrix} -8 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -8 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -6 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & -7 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -5 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -6 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -10 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -6 \end{pmatrix}$$

and

$$\Gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

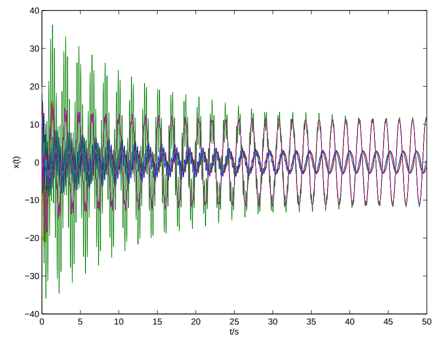

Fig.1. The dynamics of all agents $x_i(t)$.

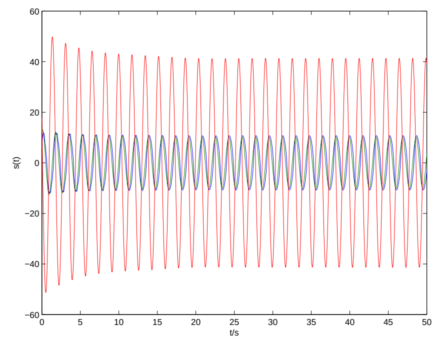

Fig.2. The average state trajectory $s(t)$.

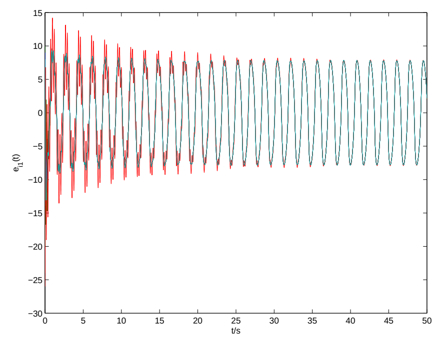

Fig.3. The consensus error dynamics $e_{i1}(t)$.

respectively.
Given the initial values of 11 agents as $(10\ 5\ -10)^T$, $(12\ 6\ -12)^T$, $(14\ 7\ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$ and $(-10\ 15\ 10)^T$ respectively, then we may verify the conditions of Theorem 3.1 readily. This demonstrates the consensus of the MAS is achieved for the time varying delay $\tau(t) > 0$. Simulation results are depicted in Fig.1 to Fig.5 for $\tau(t) = \frac{\pi}{2} + arctan(t)$ and $c = 1$. The conditions of the Theorem 4.1 are also satisfied, and the maximum allowable delay is estimated as 0.061 which reflects the conservatism of the Theorem 4.1 indirectly.
The simulation curves in Fig.1 show that the states of all agents are ultimately bounded stable. The average state

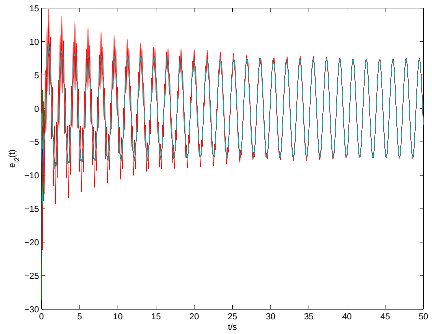

Fig.4. The consensus error dynamics $e_{i2}(t)$.

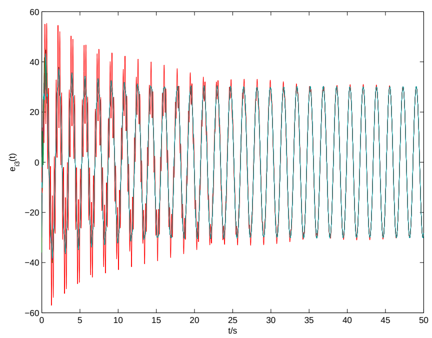

Fig.5. The consensus error dynamics $e_{i3}(t)$.

trajectory $s(t)$ is chosen as the desired moving trajectory and is depicted in Fig.2. Fig.3 to Fig.5 demonstrate that the state errors between each agent's states and the desired state trajectory respectively, and the deviation systems are also ultimately bounded stable. These simulation curves show that all agents eventually move with the desired state trajectory in the sense of boundedness.

6. Conclusions

In this paper, the consensus problems of MAS with different node dynamics and time-varying communication delay have been investigated. The derived criteria are verified via both theoretical analysis and numerical simulation. Consensus has been achieved based on a series of transformations and Lyapunov stability theorem. The consensus criteria presented here have several distinct features. Firstly, the conditions of the criteria are relatively simple in form, but are more effective to resolve the consensus problem of MAS with non-identical node dynamics. Secondly, the communication connection between agents are not direct and all delays in the communication channels are variable with time. At last, it should be noted that the conditions are still restrictive, especially for the getting of $P_i(t)$, at the same time, the delays existing in the different communication channels are assumed to be the same function and this is not always the case in practice however. Further investigations will focus on relaxing these limitations.

References

[1] Sandro Zampieri. Trends in networked control systems[C], *Proceedings of 17th World Congress of IFAC*, Seoul, Korea, July 6-11, 2008, pp. 2886 – 2894.

[2] J.P. Desai, J.P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust[J]. *IEEE Int. Trans. on Robotics and Automation*, 2001, 17(6): 905 – 908.

[3] H. Yamaguchi, T. Arai and G. Beni. A distributed control scheme for multiple robotic vehicles to make group formations[J]. *Robitics and Autonomous Systems*, 2001, 36: 125-147.

[4] Z.P. Wu, Z.H. Guan, T. Li. Group motion control of multi-agent systems based on complex network[C], *Proceedings of the 26th Chinese Control Conference*, Hunan, China, July 26-31, 2007, pp: 63 – 67.

[5] M. Porfiri, D.G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach[J]. *Automatica*, 2007, 43: 1318 – 1328.

[6] Z.K. Li, Z.S. Duan, G.R. Chen and L. Huang. Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint[J]. *IEEE Transactions on Circuits and Systems-I*, 2010, 57(1): 213 – 224.

[7] J. Cortés. Global formation-shape stabilization of relative sensing networks[C]. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 1460 – 1465.

[8] Y. Liu, Y.M. Jia, J.P. Du and S.Y. Yuan. Dynamic output feedback control for consensus of multi-agent systems: an H_∞ approach[C]. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 4470 – 4475.

[9] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays[J]. *IEEE Transactions on Automatic Control*, 2008, 53(8): 1804 – 1816.

[10] T. Li and J.F. Zhang. Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash equilibria[J]. *Automatica*, 2008, 44: 713 – 725.

[11] E.S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information[J]. *Automatica*, 2008, 44: 2766 – 2777.

[12] H.B. Jiang, J.J. Yu and C.G. Zhou. Consensus of multi-agent linear dynamic systems via impulsive control protocols[J]. *International Journal of Systems Science*, 2011, 42(6): 967 – 976.

[13] R. Olfati-Saber and R.M. Murray. Consensus problmes in networks with switching topology and time-delays[J]. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1520 – 1533.

[14] M. Arack. Passivity as a design tool for group coordination[J]. *IEEE Transactions on Automatic Control*, 2007, 52(8): 1380 – 1390.

[15] F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu and Z.Z. Yuan. Reaching a consensus via pinning control[J]. *Automatica*, 2009, 45: 1215 – 1220.

[16] Y.P. Tian and C.L Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations[J]. *Automatica*, 2009, 45: 1347 – 1353.

[17] P. Lin, Y.M. Jia and L. Li. Distributed robust H_∞ Consensus control in directed networks of agents with time-delay[J]. *Systems and Control Letters*, 2008, 57(8): 643 – 653.

[18] P.A. Bliman and G.F. Trecate. Average consensus problems in networks of agents with delayed communications[J]. *Automatica*, 2008, 44: 1985 – 1995.

[19] J. Cortes. Distributed algorithms for reaching consensus on general functions[J]. *Automatica*, 2008, 44(3): 726 – 737.

[20] D. J.Hill and J. Zhao. Global synchronization of complex dynamical networks with non-identical nodes[C], *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 9-11, 2008, 817 – 822.

[21] J. Zhao, D. J. Hill and T. Liu. Synchronization of Dynamical Networks With Nonidentical Nodes: Criteria and Control[J]. *IEEE Tran.Circuits and Syst.I*, 2011, 58(3): 584 – 594.

[22] C.C. Hua, X.P. Guan and P. Shi. Decentralized robust model reference adaptive cotnrol for interconnected time-delay systems[J]. *J.Comput.Appl.Math.*, 2006, 193: 383 – 396.

[23] Y.M. Moon, P. Park, W.H. Kwon, Y.S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems[J]. *Int J. Control*, 2001, 74: 1447 – 1455.

Proceedings of the $16^{\mathtt{th}}$ International Conference on Automation & Computing, University of Birmingham, Birmingham, UK, 11 **September** 2010

:

Global Bounded Controlled Consensus of Networked Multi-Agents Systems with Non-Identical Dynamical Agents

Wei-Song Zhong , Guo-Ping Liu, David Rees and Clive Thomas

The Faculty of Advanced Technology
University of Glamorgan
Treforest, Pontypridd, U.K., CF37 1DL
E-mail: (wzhong, gpliu, drees, ejcthomas)@glam.ac.uk

***Abstract*—This paper investigates the global bounded consensus problem of networked Multi-Agent Systems (MAS) exhibiting nonlinear, non-identical node dynamics with communication time-delays. Globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived. The proposed consensus criteria ensures that all agents eventually move along desired trajectories in terms of boundedness. The proposed controlled consensus criteria generalizes the case of identical agent dynamics to the case of non-identical agent dynamics, and many related results of other researches in this area can be viewed as special cases of the above results. We finally demonstrate the effectiveness of the theoretical results by means of a numerical simulation.**

***Index Terms*—Networked control systems, multi-agent systems, consensus, pinning control, adaptive pinning control.**

I. Introduction

Networked Multi-Agent Systems (NMAS) which deals with the study of how network architecture and interactions between network components influence global control goals, has attracted many attention due to the broad applications of NMAS in many areas. The research in this field can be categorized into two areas. One area is to deal with the design of distributed estimation techniques which can be applied to the sensor networks, and the other area is with the control of mobile autonomous agents i.e., each agent autonomously works by using information over the network from other agents [1]. How to design appropriate protocols and algorithms such that the set of agents can realize common objective, such as consensus, is a critical problem, especially for the case of unreliable information exchange and communication delays, and some relevant important contributions have been made in recent years [2∼5].

The consensus problem requires an agreement to be reached that depends on the state of all agents. The topic has been studied across many fields of science and engineering [6∼18]. It is noted that the agent dynamics in most existing works are often restricted to linear and identical ones. Obviously, in practice, this is not always the case. The controlled consensus problem of NMAS with nonlinear agent dynamics and communication delay are more complicated and just a few results have been made [19], [20]. In addition, most research in consensus problems usually assume that the final consensus value to be a constant, which may not be the case in the sense that the information state of each agent may be dynamically evolving in time according to some inherent dynamics. It is interesting to study controlled consensus problems where the final consensus value evolves with time or as a function of environmental dynamics.

The present paper will focus on the global consensus problems of NMAS based on pinning control methods [21∼24], and the proposed controlled consensus property is formulated in terms of certain boundedness of state errors. Compared with existing related results, this paper make two significant advances. One is that we generalize the related results for the case of identical agent dynamics to the case of non-identical agent dynamics, and the other is we introduce pinning controllers to the selected agents.

The rest of this paper is organized as follows. A controlled continuous-time NMAS model with communication time-delay is presented in Section II. The main results including pinning control and adaptive pinning control bounded consensus criterion are derived in Section III and IV respectively. Section V gives a numerical simulation example to verify the effectiveness of the proposed results, followed by conclusions in Section VI.

II. Problem description

Let $G = (\mathcal{V}, \mathcal{A})$ be a graph of order N consisting of a set of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and a set of edges $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$. An edge (v_j, v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{A}\}$.

We consider a NMAS consisting of N non-identical agents with communication delay:

$$\dot{x}_i = f_i(x_i) + c \sum_{j \in \mathcal{N}_i} a_{ij} \Gamma x_j(t-\tau), i = 1, 2, \cdots, N, \tag{1}$$

where $x_i = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i) : R^n \to R^n$ are continuously

Wei-Song Zhong is the corresponding author and the presenter, wzhong@glam.ac.uk

differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n\times n}$ is the inner coupling matrix, and where $\gamma_{ij} \neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N\times N}$ (which is symmetric and irreducible) represents the communication topology relation of the NMAS, and is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j\neq i} a_{ij}$. τ is a constant coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field $\bar{f}(x(t)) = \frac{1}{N}\sum_{k=1}^{N} f_k(x(t))$ with Jacobian $D\bar{f}_i(x(t))$.

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t). \tag{2}$$

We now discuss the problem of global consensus for the system (1). The consensus problem formulation in the present paper is quite different from many others, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. The consensus problem here will be depicted instead via certain boundedness of $x_i(t) - x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. This better reflects reality as it is impossible for NMAS (1) to achieve exact consensus. To address this case we will focus on making the states of all agents converge to a bounded set.

Definition 1([25], [26]): The solution $x_i(t, t_0, \psi_i)$ of the NMAS model (1) is said to be uniformly ultimately bounded with respect to the bound ε if for each $\delta > 0$ there exists $T = T(\varepsilon, \delta) > 0$ independent of t_0 such that $\|x_i(t, t_0, \psi_i)\| \leq \varepsilon$ for all $t \geq t_0 + T$ when $\|x_i(t_0)\| < \delta$, where ψ_i is the initial value.

Lemma 1 ([20]): Let $g(t)$ be a non-negative bounded function defined on R^+ and

$$\Omega = \{x(t) \in R^n | \|x(t)\| \leq \overline{lim}_{t\to\infty} g(t)\}. \tag{3}$$

Suppose there exists a strictly positive definite matrix $P(t) \in \mathcal{PC}^1_{n\times n}$ and a constant $\delta > 0$ such that the derivative of $V(x(t), t) = x^T(t)P(t)x(t)$ along the trajectory of the system

$$\dot{x}(t) = f(x(t), t), \quad x(t) \in R^n, t \in [0, \infty) \tag{4}$$

satisfies

$$\dot{V} \leq -\delta\|x(t)\|^2 \quad \texttt{if} \quad \|x(t)\| \geq g(t). \tag{5}$$

For any $t > 0$, let

$$Q_t = \{x(t) | V(x(t), t) \leq \sup_{y(s)\in\Omega, s\geq 0} \{V(y(s), s)\}\} \tag{6}$$

and

$$c = \overline{lim}_{t\to\infty}(max\{\|x(t)\| | x(t) \in Q_t\}). \tag{7}$$

Then, $x(t)$ converges to the set

$$M = \{x(t) | \|x(t)\| \leq c\}. \tag{8}$$

We denote $x(t)$, $s(t)$, $u(t)$, $e(t)$, $w(t)$, $d_i(t)$ and $V(w(t), t)$ as x, s, u, e, w, d_i and V respectively.

III. Linear feedback pinning controller

To achieve the goal, we apply the feedback control strategy on a small fraction δ $(0 < \delta \leq 1)$ of the agents in system (1). Suppose that nodes $i_1, i_2, \cdots, i_l$ are selected to be under control, where $l = [\delta N]$ stands for the smaller but nearest integer to the real number δN. This controlled NMAS can be described as

$$\begin{cases} \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j\in\mathcal{N}_i} a_{i_k j}\Gamma x_j(t-\tau) + u_{i_k}, 1 \leq k \leq l, \\ \dot{x}_{i_k} = f_{i_k}(x_{i_k}) + c\sum_{j\in\mathcal{N}_i} a_{i_k j}\Gamma x_j(t-\tau), l+1 \leq k \leq N. \end{cases} \tag{9}$$

The local linear negative feedback control law is chosen as follows:

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k} - s), & 1 \leq k \leq l, \\ u_{i_k} = 0, & l+1 \leq k \leq N, \end{cases} \tag{10}$$

where the feedback gain $d_{i_k} > 0$.

Combine (9) and (10) and rearrange the order of the nodes in the network. Let the first l nodes be controlled, and $e_i = x_i - s$, $i = 1, 2, \cdots, N$. It's obvious that $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i} a_{kj}\Gamma x_j(t-\tau) = 0$ and $\sum_{i=1}^{N} e_i = 0$. Then by applying the Newton-Leibniz formula, error systems can be written as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau) \\ \quad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad + f_i(s) - \bar{f}(s) - d_i e_i, & 1 \leq i \leq l, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau) \\ \quad + \int_0^1 (Df_i + \tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad - \frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad + f_i(s) - \bar{f}(s), & l+1 \leq i \leq N. \end{cases} \tag{11}$$

The following work will focus on simplifying the error systems (11) by means of a series of transformations using a procedure similar to [20].

Define the following matrix

$$D = diag(D_1, D_2, \cdots, D_N) \in R^{nN\times nN},$$

where $D_i = diag\{-d_i, -d_i, \cdots, -d_i\} \in R^{n\times n}$.

Let $e = (e_1^T, e_2^T, \cdots, e_N^T)^T$, then (11) becomes

$$\dot{e} = \bar{\Sigma}(t)e + cA \otimes \Gamma e(t-\tau) + I(t)e - \frac{1}{N}H(t)e + F(t), \tag{12}$$

where $I(t) = diag\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau \cdots \int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\}$, $\bar{\Sigma}(t) = I_N \otimes D\bar{f}(s) + D$, $H^T(t) = (H_1^T(t), \cdots, H_N^T(t))$, $H_i(t) = (\int_0^1 Df_1(s + \tau e_1)d\tau, \cdots, \int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t) = (f_1^T(s) - \bar{f}^T(s), \cdots, f_N^T(s) - \bar{f}^T(s))$.

Since A is symmetric and irreducible, according to [20], there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} =$

$(\Phi_1, \Phi_2, \cdots, \Phi_N)$. This together with $w(t) = (\Phi^T \otimes I_n)e$ gives

$$\begin{aligned}\dot{w} &= (\Phi^T \otimes I_n)\bar{\Sigma}(t)(\Phi \otimes I_n)w \\ &+ (\Phi^T \otimes I_n)(cA \otimes \Gamma)(\Phi \otimes I_n)w(t-\tau) \\ &+ (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ &- \frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n)w + (\Phi^T \otimes I_n)F(t).\end{aligned} \tag{13}$$

Note that $H(t) = \sqrt{N}\sum_{k=1}^{N}(\mathbf{0} \ \cdots \ \mathbf{0} \ \bar{\Phi}_k \ \mathbf{0} \ \cdots \ \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its k-th column equal to Φ_1 and the remaining elements are zero. Then we have $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}(\mathbf{0} \cdots \mathbf{0} \, I_k \, \mathbf{0} \cdots \mathbf{0}) \otimes \int_0^1 Df_k(s+\tau e_k)d\tau(\Phi \otimes I_n)$, where I_k stands for the matrix with its k-th column equals $(1 \ 0 \ \cdots \ 0)^T$ and the remaining of its elements are zero.

Thus, a simple calculation gives $\frac{1}{N}(\Phi^T \otimes I_n)H(t)(\Phi \otimes I_n) = \frac{1}{\sqrt{N}}\sum_{k=1}^{N}\begin{pmatrix}\Upsilon_k \\ 0\end{pmatrix} \otimes \int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k \in R^{1\times N}$ and $0 \in R^{(N-1)\times N}$. Therefore, $\dot{w} = \bar{\Sigma}(t)w + c\Lambda \otimes \Gamma w(t-\tau) + (\Phi^T \otimes I_n)I(t)(\Phi \otimes I_n)w - \begin{pmatrix} * \\ 0\end{pmatrix} w + (\Phi^T \otimes I_n)F(t)$. Since $w_1 \equiv 0$, we only need to consider $w_2, w_3, \cdots, w_N$. Rewriting in the component form we have

$$\begin{aligned}\dot{w}_i &= \Sigma_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau) + (\Phi_i^T \otimes I_n)F(t) \\ &+ (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w, \ i = 2, 3, \cdots, N,\end{aligned} \tag{14}$$

where $\Sigma_i = \bar{D}f(s) + D_i$.

So far, we have transferred the consensus problem of system (1) to the stability problem of the $N-1$ of $n-$dimensional systems.

Theorem 1 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$\begin{aligned} a\|x\|^2 \leq x^T P_i(t)x + \int_{t-\tau}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \leq b\|x\|^2, \\ \forall t \in R^+, \ x \in R^n, i = 2, 3, \cdots, N,\end{aligned} \tag{15}$$

$$\begin{aligned}&\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i \\ &+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \zeta I \leq 0, \ i = 1, 2, \cdots, N,\end{aligned} \tag{16}$$

$$\|I(t)\| \leq \gamma, \ i = 1, 2, \cdots, N. \tag{17}$$

Let

$$\mu(t) = \|F(t)\| \tag{18}$$

be bounded and

$$\beta = (\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}}, \tag{19}$$

if $\zeta > 2\gamma\beta$, then system (12) converges to the set

$$M = \{e \| \|e\| \leq \frac{2b}{a}\frac{\beta\overline{lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\}, \tag{20}$$

for any fixed time delay $\tau > 0$, namely, $e(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^{N}x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, and then the NMAS (1) achieves bounded consensus for any fixed time delay $\tau > 0$.

Proof. Choose the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N} V_i, \tag{21}$$

$$V_i = w_i^T P_i(t)w_i + \int_{t-\tau}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha. \tag{22}$$

Differentiating (22) along the trajectory of (14) gives

$$\begin{aligned}\dot{V}_i &= w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i)w_i \\ &+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \\ &+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) + 2w_i^T(c\lambda_i P_i(t)\Gamma)w_i(t-\tau) \\ &- w_i^T(t-\tau)Q_i w_i(t-\tau).\end{aligned} \tag{23}$$

Applying the Young Inequality to the equality (23) results in

$$\begin{aligned}\dot{V}_i &\leq w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i \\ &+ c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) \\ &+ 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w.\end{aligned} \tag{24}$$

Condition (16) implies that the first term on the right hand side of (24) satisfies

$$\begin{aligned}&w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i \\ &\quad + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i \leq -\zeta\|w_i\|^2.\end{aligned} \tag{25}$$

The second term on the right hand side of (24) satisfies

$$2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i\|. \tag{26}$$

Applying condition (17) we know the third term on the right hand side of (24) satisfies

$$2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \leq 2\gamma\|P_i(t)\|\|w_i\|\|w\|. \tag{27}$$

Since $V = \sum_{i=2}^{N} V_i$, we have

$$\begin{aligned}\dot{V} &= \sum_{i=2}^{N}\dot{V}_i \\ &= -\zeta\|w\|^2 + 2(\gamma\|w\| + \mu(t))\sum_{i=2}^{N}\|w_i\|\|P_i(t)\| \\ &\leq -\zeta\|w\|^2 + 2(\gamma\|w\| + \mu(t))\|w\|(\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}} \\ &= \|w\|((2\gamma\beta - \zeta)\|w\| + 2\beta\mu(t)).\end{aligned} \tag{28}$$

Thus, when

$$\|w\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \tag{29}$$

we have

$$\dot{V} \leq -\delta\|w\|^2. \tag{30}$$

Applying Lemma 1 completes the proof.

IV. Adaptive pinning controller

In this section, we will derive globally consensus criteria via direct adaptive pinning control method. Without loss of generality, we still assume that the first l agents are selected as pinned agents with the adaptive controllers:

$$\begin{cases} u_i = -d_i(x_i - s), & 1 \le i \le l, \\ \dot{d}_i = h_i e_i^T P_i(t) e_i, \\ u_i = 0, & l+1 \le i \le N, \end{cases} \tag{31}$$

where constant $h_i > 0$ and positive definite matrix $P_i(t) \in R^{n\times n}$. Applying Newton-Leibniz formula, then the error NMAS can be rewritten as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau) \\ \quad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad +f_i(s) - \bar{f}(s) - d_i e_i, \qquad 1 \le i \le l, \\ \dot{d}_i = h_i e_i^T P_i(t) e_i, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau) \\ \quad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \quad -\frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \quad +f_i(s) - \bar{f}(s), \qquad l+1 \le i \le N. \end{cases} \tag{32}$$

Repeating a similar procedure to the previous subsection, the controlled consensus problem of system (1) is equivalent to the stability problem of the following $N-1$ of n-dimensional systems.

$$\begin{cases} \dot{w}_i = D\bar{f}(s(t))w_i - d_i w_i + c\lambda_i\Gamma w_i(t-\tau) \\ \quad +(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \quad +(\Phi_i^T \otimes I_n)F(t), \qquad 2 \le i \le l, \\ \dot{d}_i = h_i w_i^T P_i(t) w_i, \\ \dot{w}_i = D\bar{f}(s)w_i + c\lambda_i\Gamma w_i(t-\tau) \\ \quad +(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \quad +(\Phi_i^T \otimes I_n)F(t), \qquad l+1 \le i \le N, \end{cases} \tag{33}$$

where w_i, w, Φ, Φ_i, $I(t)$ and $F(t)$ are the same as the previous subsection.

Theorem 2 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\bar{\zeta} > 0$, $\gamma \ge 0$, $a > 0$ and $b > 0$ such that

$$a\|x\|^2 \le x_i^T P_i(t) x_i + \int_{t-\tau}^t x_i^T(\alpha)Q_i x_i(\alpha)d\alpha + \frac{(d_i-d)^2}{h_i} \le b\|x\|^2, \forall t \in R^+,\ x \in R^n, i = 2,3,\cdots,N, \tag{34}$$

$$\dot{P}_i(t) + P_i(t)D\bar{f}(s) + (D\bar{f}(s))^T P_i(t) + Q_i - 2dP_i(t) + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \bar{\zeta} I \le 0,\ i = 1,2,\cdots,N, \tag{35}$$

(17) and $\bar{\zeta} > 2\gamma\beta$ are satisfied, then the system (12) converges to the set (20) for any fixed time delay $\tau > 0$, where $\mu(t)$ and β are the same as in (18) and (19) respectively, $\bar{\delta} > 0$ is any constant satisfying $\bar{\delta} < \bar{\zeta} - 2\gamma\beta$, and then the NMAS (1) achieves bounded consensus for any fixed time delay $\tau > 0$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^N V_i + \sum_{i=2}^l \frac{(d_i-d)^2}{h_i}, \tag{36}$$

where

$$\begin{cases} V_i = w_i^T P_i(t) w_i + \int_{t-\tau}^t w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \\ \quad + \frac{(d_i-d)^2}{h_i}, & 2 \le i \le l, \\ V_i = w_i^T P_i(t) w_i + \int_{t-\tau}^t w_i^T(\alpha)Q_i w_i(\alpha)d\alpha, & l+1 \le i \le N, \end{cases} \tag{37}$$

where d is a positive constant to be determined.

Differentiating (37) along the trajectory of (33) gives

$$\begin{aligned} \dot{V}_i = {} & w_i^T(\dot{P}_i(t) + P_i(t)D\bar{f}(s) + (D\bar{f}(s))^T P_i(t) + Q_i \\ & - 2dP_i(t))w_i + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \\ & + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) + 2w_i^T(c\lambda_i P_i(t)\Gamma)w_i(t-\tau) \\ & - w_i^T(t-\tau)Q_i w_i(t-\tau). \end{aligned} \tag{38}$$

The remaining part of the proof is similar to that of Theorem 1, so is therefore omitted here. This completes the proof.

V. Examples

To demonstrate the theoretical results obtained above, we construct a NMAS consisting of 12 agents described as follows

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma x_j(t-\tau), \tag{39}$$

where $f_i(x_i(t)) = B_i x_i(t) + g(x_i(t))$, $B_i(i = 1,2,\cdots,6)$ and $B_i(i = 7,8,\cdots,12)$ are chosen as follows:

$$\begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix},$$

$$\begin{pmatrix} -10-0.1\times(i-6) & 10+0.1\times(i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15+0.1\times(i-6) & 0 \end{pmatrix},$$

and

$$g(x_i(t)) = (-9.5sin(\frac{\pi x_{i1}(t)}{3.2} + \pi)\ 0\ 0)^T, \quad i = 1,2,\cdots,12.$$

The communication coupling matrix $C = (C_1^T C_2^T \cdots C_{12}^T)$, $C_1 = (-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0)$, $C_2 = (1\ -8\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0)$, $C_3 = (1\ 1\ -7\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1)$, $C_4 = (0\ 1\ 1\ -6\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1)$, $C_5 = (1\ 1\ 0\ 0\ -6\ 0\ 1\ 1\ 1\ 1\ 0\ 0)$, $C_6 = (1\ 0\ 0\ 1\ 0\ -5\ 1\ 0\ 1\ 1\ 0)$, $C_7 = (1\ 1\ 0\ 1\ 1\ 1\ -7\ 1\ 0\ 1\ 0)$, $C_8 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ -6\ 0\ 1\ 1\ 1)$, $C_9 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ -7\ 1\ 1\ 1)$, $C_{10} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ -10\ 1)$, $C_{11} = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ -7\ 1)$, $C_{12} = (0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ -5)$. $\Gamma = diag\{2,2,2\}$, respectively, where the matrix A is produced by means of the Scale-Free network program.

Design the following controllers

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k}(t) - s(t)), & i_k = 1,2 \texttt{ and } 10, \\ u_{i_k} = 0, & \texttt{else}, \end{cases}$$

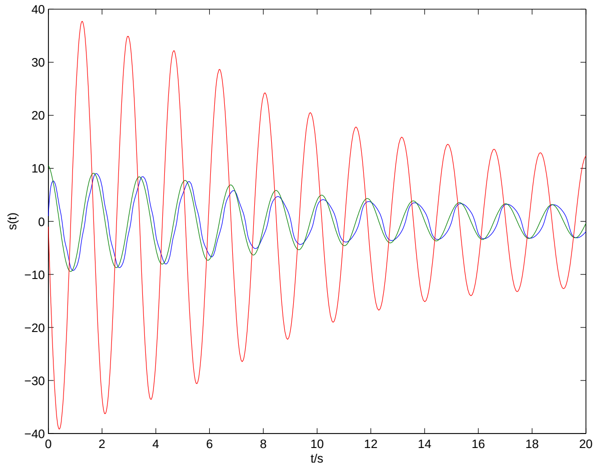

Fig.1. Desired agent dynamics under pinning control.

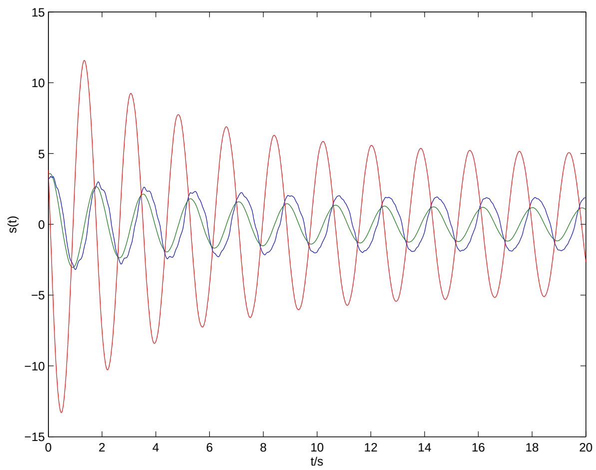

Fig.2. Desired agent dynamics under adaptive pinning control.

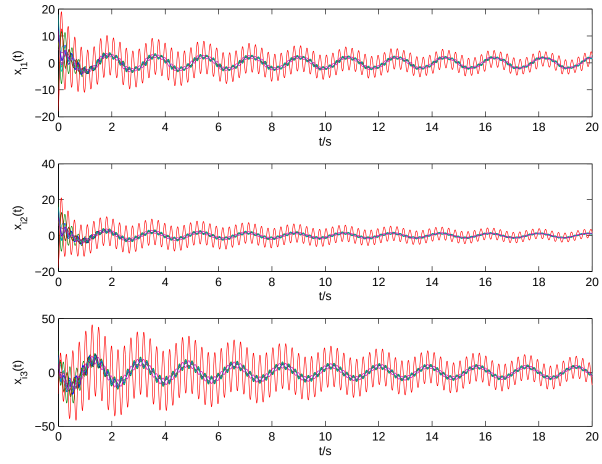

Fig.3. All agent dynamics under pinning control.

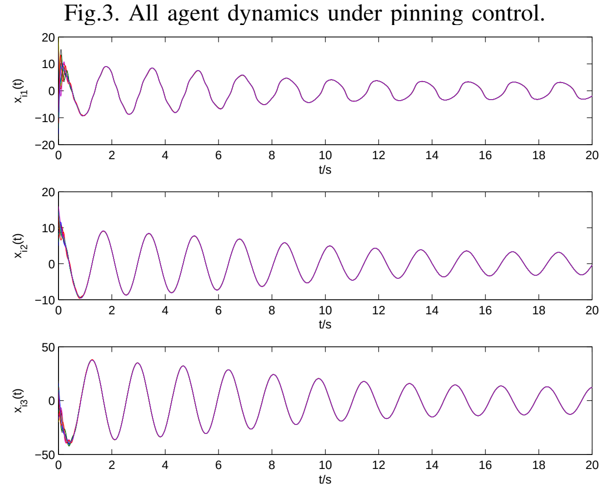

Fig.4. All agent dynamics under adaptive pinning control.

with $d_1 = 0.5$, $d_2 = 0.5$, $d_{10} = 0.5$ and

$$\begin{cases} u_{i_k} = -d_{i_k}(t)(x_{i_k}(t) - s(t)), & i_k = 1, 2 \text{ and } 10, \\ \dot{d}_{i_k}(t) = h_{i_k} e_{i_k}^T P_{i_k}(t) e_{i_k}, & \\ u_{i_k} = 0, & \text{else}, \end{cases}$$

with $h_1 = 0.1$, $h_2 = 0.2$, $h_{10} = 0.3$, $s(t)$ can then be evaluated by simulation.

Given the initial values of 12 agents as $(10\ 5\ \ -10)^T$, $(12\ 6\ \ -12)^T$, $(14\ 7\ \ -14)^T$, $(16\ 8\ \ -16)^T$, $(18\ 9\ \ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$, $(-10\ 15\ 10)^T$, $(-8\ 16\ 8)^T$ respectively and $P_{i_k}(t) = I_3$, $d_1(0) = 1$, $d_2(0) = 1$, $d_{10}(0) = 1$. We may verify the conditions of Theorem 1 and Theorem 2 readily. This demonstrates the bounded consensus of the NMAS is achieved for any time delay $0 < \tau \leq 0.06$. Simulation results are depicted in Fig.1 to Fig.8 for $\tau = 0.06$ and $c = 1$.

VI. Conclusion

In this paper, we've investigated the controlled consensus problems of NMAS with different agent dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on pinning control and adaptive pinning control methods. It should be noted that the conditions are still restrictive and all the delays are the same. Further investigations will focus on relaxing these limitations.

VII. Acknowledgment

The authors would like to thank the anonymous reviewers for their valuable suggestions and comments.

References

[1] S. Zampieri, "Trends in networked control systems,"Proceedings of 17th World Congress of IFAC. Seoul, Korea, pp. 2886-2894, July 6-11 2008.

[2] J.P. Desai, J.P. Ostrowski, and V. Kumar, "Modeling and control of formation of nonholonomic mobile robust,"IEEE Int. Trans. on Robotics and Automation. vol. 17, pp. 905-908, 2001.

[3] W. Ren, R.W. Beard, "Formation feedback control for multiple spacecraft via virtual structures,"IEE Proc. Control Theory and Applications. vol. 151, pp. 357-368, 2004.

[4] M. Porfiri, D.G. Roberson, and D.J Stilwell, "Tracking and formation control of multiple autonomous agents: a two-level consensus approach, "Automatica. vol. 43, pp. 1318-1328, 2007.

[5] J. Cortés, "Global formation-shape stabilization of relative sensing networks,"2009 American Control Systems. Hyatt Regency Riverfront, St. Louis, USA, pp. 1460-1465, June 10-12 2009.

[6] Y. Liu, Y.M. Jia, J.P. Du, and S.Y. Yuan, "Dynamic output feedback control for consensus of multi-agent systems: an H_∞ approach,"2009 American Control Systems. Hyatt Regency Riverfront, St. Louis, USA, pp. 4470-4475, June 10-12 2009.

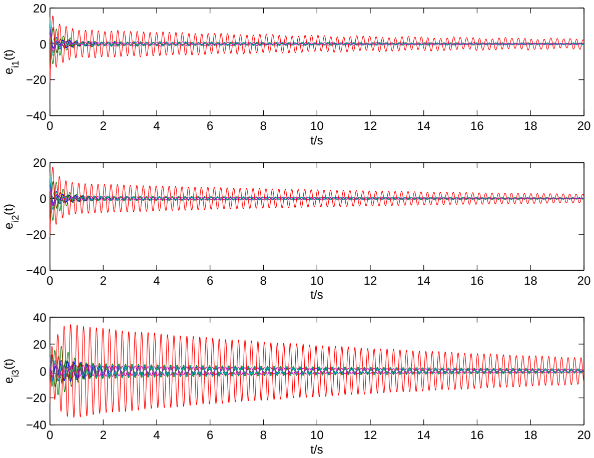

Fig.5. All agent error dynamics under pinning control.

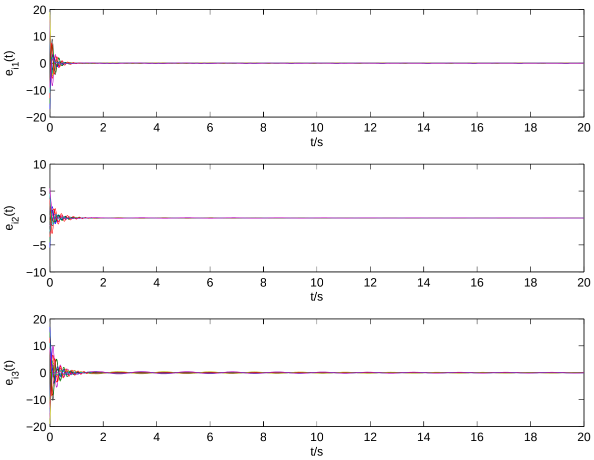

Fig.6. All agent error dynamics under adaptive pinning control.

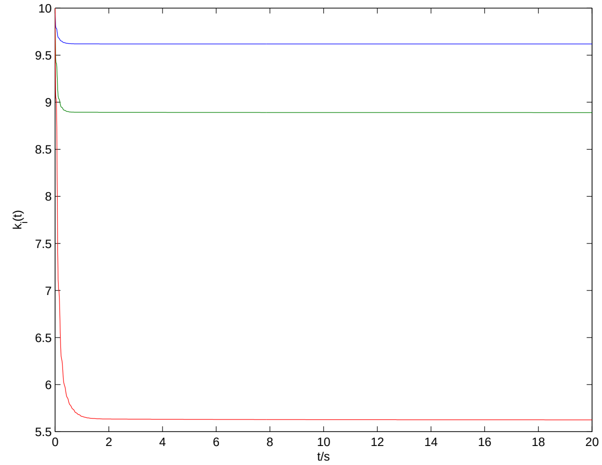

Fig.7. Adaptive gain curves.

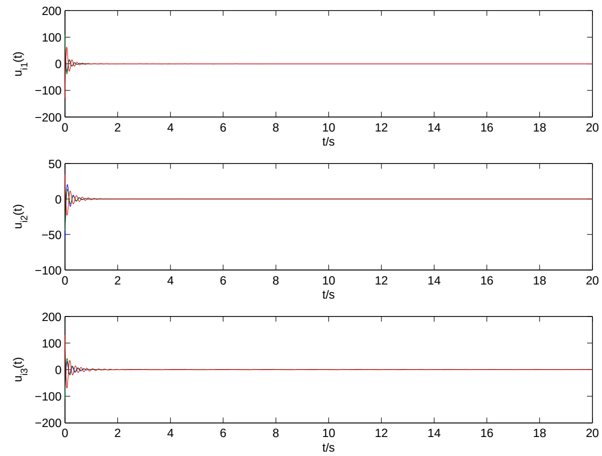

Fig.8. Adaptive pinning controllers curves.

[7] Y.G. Hong, J.P. Hu, and L.X. Gao, "Tracking control for multi-agent consensus with an active leader and variable topology,"Automatica. vol. 42, pp. 1177-1182 2006.

[8] F. Xiao and L. Wang, "Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays,"IEEE Transactions on Automatic Control. vol. 53, pp. 1804-1816 2008.

[9] T. Li and J.F. Zhang, "Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash equilibria,"Automatica. vol. 44, pp. 713-725 2008.

[10] E.S. Kazerooni, and K. Khorasani, "Optimal consensus algorithms for cooperative team of agents subject to partial information,"Automatica. vol. 44, pp. 2766-2777 2008.

[11] Z.K. Li, Z.S. Duan, G.R. Chen, and L. Huang, "Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint,"IEEE Transactions on Circuits and Systems-I. vol. 57, pp. 213-224 2010.

[12] R. Olfati-Saber, and R.M. Murray, "Consensus problmes in networks with switching topology and time-delays,"IEEE Transactions on Automatic Control. vol.49, pp. 1520-1533 2004.

[13] M. Arack, "Passivity as a design tool for group coordination,"IEEE Transactions on Automatic Control. vol. 52, pp. 1380-1390 2007.

[14] F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu, and Z.Z. Yuan, "Reaching a consensus via pinning control,"Automatica. vol. 45, pp. 1215-1220 2009.

[15] Y.P. Tian, and C.L Liu, "Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations,"Automatica. vol. 45, pp. 1347-1353 2009.

[16] P. Lin, Y.M. Jia, and L. Li, "Distributed robust H_∞ Consensus control in directed networks of agents with time-delay,"Systems and Control Letters. vol. 57, pp. 643-653 2008.

[17] P.A. Bliman, and G.F. Trecate, "Average consensus problems in networks of agents with delayed communications,"Automatica. vol. 44, pp. 1985-1995 2008.

[18] J. Cortes, "Distributed algorithms for reaching consensus on general functions,"Automatica. vol. 44, pp. 726-737 2008.

[19] Z.S Duan, and G.R. Chen, "Global Robust Stability and Synchronization of Networks With Lorenz-Type Nodes,"IEEE Transactions on Circuits and Systems-II: Express Briefs. vol. 56, pp. 679-683 2009.

[20] D. J.Hill, and J. Zhao, "Global synchronization of complex dynamical networks with non-identical nodes,"Proceedings of the 47th IEEE Conference on Decision and Control. Cancun, Mexico, pp. 817-822, Dec. 9-11, 2008.

[21] J.C. Zhao, J.A. Lu and Q.J. Zhang, "Pinning a complex dynamical network to a homogenous trajectory,"IEEE Transactions on Circuits and Systems-II: Express Briefs. vol. 56, pp. 514-518 2009.

[22] X. Li, X.F. Wang, and G.R. Chen, "Pinning a complex dynamical network to its equilibrium,"IEEE Tran.Circuits and Syst.I. vol. 51, pp. 2074-2087 2004.

[23] W. Wu, W.J. Zhou, and T.P. Chen, "Cluster Synchronization of Linearly Coupled Complex Networks Under Pinning Control,"IEEE Transactions on Circuits and Systems-I. vol. 56, pp. 829-839 2009.

[24] Y.Y. Wu, W. Wei, G.Y. Li, and J. Xiang, "Pinning Control of Uncertain Complex Networks to a Homogeneous Orbit,"IEEE Transactions on Circuits and Systems-II: Express Briefs. vol 56, pp. 235-239 2009.

[25] Y.M. Moon, P. Park, W.H. Kwon, Y.S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems,"Int J. Control. vol 74, pp. 1447-1455 2001.

[26] C.C. Hua, X.P. Guan, and P. Shi, "Decentralized robust model reference adaptive cotnrol for interconnected time-delay systems,"J.Comput.Appl.Math. vol 193, pp. 383-396 2006.

:

Global Controlled Consensus of Multi-Agent Systems with Different Agent Dynamics and Time-Varying Communication Delay

Wei-Song Zhong
Faculty of Advanced Technology
University of Glamorgan
Cardiff, UK CF37 1DL
Email: wzhong@glam.ac.uk

Guo-Ping Liu
Faculty of Advanced Technology
University of Glamorgan
Cardiff, UK CF37 1DL
Email: gpliu@glam.ac.uk

Clive Thomas
Faculty of Advanced Technology
University of Glamorgan
Cardiff, UK CF37 1DL
Email: ejcthomas@glam.ac.uk

***Abstract*—This paper investigates the global bounded consensus problem of Networked Multi-Agent Systems exhibiting nonlinear, non-identical agent dynamics with communication time-varying delay. Globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived. The proposed consensus criteria ensures that all agents eventually move along desired trajectories in terms of boundedness. The proposed controlled consensus criteria generalizes the case of identical agent dynamics to the case of non-identical agent dynamics, and many related results of other researches in this area can be viewed as special cases of the above results. We finally demonstrate the effectiveness of the theoretical results by means of a numerical simulation.**

I. Introduction

Networked Multi-Agent Systems (NMAS) has attracted many attention due to the broad applications of NMAS in many areas. How to design appropriate protocols and algorithms such that the set of agents can realize common objective, such as consensus, is a critical problem, especially for the case of unreliable information exchange and communication delays, and some relevant important contributions have been made in recent years [1], [2], [3], [4].

The consensus problem requires an agreement to be reached that depends on the state of all agents. The topic has been studied across many fields of science and engineering [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]. It is noted that the agent dynamics in most existing works are often restricted to linear and identical ones. Obviously, in practice, this is not always the case. The controlled consensus problem of NMAS with nonlinear agent dynamics and communication delay are more complicated and just a few results have been made [21], [22]. In addition, most research in consensus problems usually assume that the final consensus value to be a constant, which may not be the case in the sense that the information state of each agent may be dynamically evolving in time according to some inherent dynamics. It is interesting to study controlled consensus problems where the final consensus value evolves with time or as a function of environmental dynamics.

The behavior of the NMAS with non-identical agent dynamics is much more complicated than the identical case. Usually, no common equilibrium for all agents exists even if each agent has an equilibrium, neither does a consensus manifold exist in the classical sense. The NMAS with non-identical agent dynamics cannot be decoupled into a number of lower dimensional systems exactly like the identical-agent case. Yet, a NMAS with non-identical agents may still exhibit some kinds of consensus behaviors which are far from being fully understood. Certain reasonable and satisfactory boundedness of state motions errors between different agents can be taken as useful consensus properties. The present paper will focus on the global consensus problems of NMAS based on pinning control methods [23], [24], [25], [26], and the proposed controlled consensus property is formulated in terms of certain boundedness of state errors.

The rest of this paper is organized as follows. A controlled continuous-time NMAS model with communication time-delay is presented in Section II. The main results including pinning control and adaptive pinning control bounded consensus criterion are derived in Section III and IV respectively. Section V gives a numerical simulation example to verify the effectiveness of the proposed results, followed by conclusions in Section VI.

II. Problem description

Let $G = (\mathcal{V}, \mathcal{A})$ be a graph of order N consisting of a set of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and a set of edges $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$. An edge (v_j, v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{A}\}$.

We consider a NMAS consisting of N non-identical agents with communication delay:

$$\dot{x}_i = f_i(x_i) + c \sum_{j \in \mathcal{N}_i} a_{ij} \Gamma x_j(t - \tau(t)), i = 1, 2, \cdots, N, \quad (1)$$

where $x_i = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c>0$ denotes the coupling strength, $\Gamma=(\gamma_{ij})\in R^{n\times n}$ is the inner coupling matrix, and where $\gamma_{ij}\neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A=(a_{ij})\in R^{N\times N}$ (which is symmetric and irreducible) represents the communication topology relation of the NMAS, and is defined by $a_{ij}=a_{ji}=1(v_j\in\mathcal{N}_i)$, $a_{ij}=0(v_j\notin\mathcal{N}_i)$ and $a_{ii}=-\sum_{j\neq i}a_{ij}$. $\tau(t)$ is a time-varying coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field $\bar{f}(x(t))=\frac{1}{N}\sum_{k=1}^{N}f_k(x(t))$ with Jacobian $D\bar{f}_i(x(t))$.

The average state trajectory is chosen as the desired moving trajectory

$$s(t)=\frac{1}{N}\sum_{k=1}^{N}x_k(t). \tag{2}$$

We now discuss the problem of global consensus for the system (1). The consensus problem here will be depicted instead via certain boundedness of $x_i(t)-x_j(t)$, $\forall i,j=1,2,\cdots,N$ as $t\to\infty$. This better reflects reality as it is impossible for NMAS (1) to achieve exact consensus. To address this case we will focus on making the states of all agents converge to a bounded set.

We denote $x(t)$, $s(t)$, $u(t)$, $e(t)$, $w(t)$, $d_i(t)$ and $V(w(t),t)$ as x, s, u, e, w, d_i and V respectively.

III. Linear feedback pinning controller

To achieve the goal, we apply the feedback control strategy on a small fraction δ $(0<\delta\leq 1)$ of the agents in system (1). Suppose that nodes $i_1,i_2,\cdots,i_l$ are selected to be under control, where $l=[\delta N]$ stands for the smaller but nearest integer to the real number δN. This controlled NMAS can be described as

$$\begin{cases}\dot{x}_{i_k}=f_{i_k}(x_{i_k})+c\sum_{j\in\mathcal{N}_i}a_{i_kj}\Gamma x_j(t-\tau(t))\\ \qquad +u_{i_k}, \qquad\qquad 1\leq k\leq l,\\ \dot{x}_{i_k}=f_{i_k}(x_{i_k})+c\sum_{j\in\mathcal{N}_i}a_{i_kj}\Gamma x_j(t-\tau(t)),\\ \qquad\qquad\qquad\qquad l+1\leq k\leq N.\end{cases} \tag{3}$$

The local linear negative feedback control law is chosen as follows:

$$\begin{cases}u_{i_k}=-d_{i_k}(x_{i_k}-s), & 1\leq k\leq l,\\ u_{i_k}=0, & l+1\leq k\leq N,\end{cases} \tag{4}$$

where the feedback gain $d_{i_k}>0$.

Combine (3) and (4) and rearrange the order of the nodes in the network. Let the first l nodes be controlled, and $e_i=x_i-s$, $i=1,2,\cdots,N$. It's obvious that $\frac{c}{N}\sum_{k=1}^{N}\sum_{j\in\mathcal{N}_i}a_{kj}\Gamma x_j(t-\tau(t))=0$ and $\sum_{i=1}^{N}e_i=0$. Then by applying the Newton-Leibniz formula, error systems can be written as

$$\begin{cases}\dot{e}_i=D\bar{f}(s)e_i+c\sum_{j\in\mathcal{N}_i}a_{ij}\Gamma e_j(t-\tau(t))\\ \quad +\int_0^1(Df_i(s+\tau e_i)-D\bar{f}(s))e_i d\tau\\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau\\ \quad +f_i(s)-\bar{f}(s)-d_ie_i, \quad 1\leq i\leq l,\\ \dot{e}_i=D\bar{f}(s)e_i+c\sum_{j\in\mathcal{N}_i}a_{ij}\Gamma e_j(t-\tau(t))\\ \quad +\int_0^1(Df_i+\tau e_i)-D\bar{f}(s))e_i d\tau\\ \quad -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau\\ \quad +f_i(s)-\bar{f}(s), \quad l+1\leq i\leq N.\end{cases} \tag{5}$$

The following work will focus on simplifying the error systems (5) by means of a series of transformations using a procedure similar to [22].

Define the following matrix

$$D=diag(D_1,D_2,\cdots,D_N)\in R^{nN\times nN},$$

where $D_i=diag\{-d_i,-d_i,\cdots,-d_i\}\in R^{n\times n}$.

Let $e=(e_1^T,e_2^T,\cdots,e_N^T)^T$, then (5) becomes

$$\dot{e}=\bar{\Sigma}(t)e+cA\otimes\Gamma e(t-\tau(t))+I(t)e-\frac{1}{N}H(t)e+F(t), \tag{6}$$

where $I(t)=diag\{\int_0^1(Df_1(s+\tau e_1)-D\bar{f}(s))d\tau\cdots\int_0^1(Df_N(s+\tau e_N)-D\bar{f}(s))d\tau\}$, $\bar{\Sigma}(t)=I_N\otimes D\bar{f}(s)+D$, $H^T(t)=(H_1^T(t),\cdots,H_N^T(t))$, $H_i(t)=(\int_0^1 Df_1(s+\tau e_1)d\tau,\cdots,\int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t)=(f_1^T(s)-\bar{f}^T(s),\cdots,f_N^T(s)-\bar{f}^T(s))$.

Since A is symmetric and irreducible, according to [22], there exists a unitary matrix $\Phi=(\varphi_{ij})_{N\times N}=(\Phi_1,\Phi_2,\cdots,\Phi_N)$. This together with $w(t)=(\Phi^T\otimes I_n)e$ gives

$$\begin{aligned}\dot{w}=&(\Phi^T\otimes I_n)\bar{\Sigma}(t)(\Phi\otimes I_n)w\\ &+(\Phi^T\otimes I_n)(cA\otimes\Gamma)(\Phi\otimes I_n)w(t-\tau(t))\\ &+(\Phi^T\otimes I_n)I(t)(\Phi\otimes I_n)w+(\Phi^T\otimes I_n)F(t)\\ &-\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)w.\end{aligned} \tag{7}$$

Note that $H(t)=\sqrt{N}\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0})\otimes\int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its k-th column equal to Φ_1 and the remaining elements are zero. Then we have $\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)=\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0})\otimes\int_0^1 Df_k(s+\tau e_k)d\tau(\Phi\otimes I_n)$, where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the remaining of its elements are zero.

Thus, a simple calculation gives $\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)=\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\begin{pmatrix}\Upsilon_k & 0\end{pmatrix}^T\otimes\int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k\in R^{1\times N}$ and $0\in R^{(N-1)\times N}$. Therefore, $\dot{w}=\bar{\Sigma}(t)w+c\Lambda\otimes\Gamma w(t-\tau(t))+(\Phi^T\otimes I_n)I(t)(\Phi\otimes I_n)w-\begin{pmatrix}* & 0\end{pmatrix}^T w+(\Phi^T\otimes I_n)F(t)$. Since $w_1\equiv 0$, we only need to consider $w_2,w_3,\cdots,w_N$. Rewriting in the component form we have

$$\begin{aligned}\dot{w}_i=&\Sigma_i(t)w_i+c\lambda_i\Gamma w_i(t-\tau(t))+(\Phi_i^T\otimes I_n)F(t)\\ &+(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w,\ i=2,3,\cdots,N,\end{aligned} \tag{8}$$

where $\Sigma_i = \bar{D}f(s) + D_i$.

So far, we have transferred the consensus problem of system (1) to the stability problem of the $N-1$ of $n-$dimensional systems.

Theorem 1 Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, Q_i and constants $\zeta > 0$, $\gamma \geq 0$, $a > 0$ and $b > 0$ such that

$$a\|x\|^2 \leq x^T P_i(t)x + \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha \leq b\|x\|^2,\\ \forall t \in R^+,\ x \in R^n, i = 2,3,\cdots,N, \tag{9}$$

$$\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i\\ + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \zeta I \leq 0,\ i = 1,2,\cdots,N, \tag{10}$$

$$\|I(t)\| \leq \gamma,\ i = 1,2,\cdots,N. \tag{11}$$

Let

$$\mu(t) = \|F(t)\| \tag{12}$$

be bounded and

$$\beta = (\sum_{i=2}^{N} \|P_i(t)\|^2)^{\frac{1}{2}}, \tag{13}$$

if $\zeta > 2\gamma\beta$, then system (6) converges to the set

$$M = \{e| \|e\| \leq \frac{2b}{a}\frac{\beta \overline{lim}_{t\to\infty}\mu(t)}{\zeta - 2\gamma\beta - \delta}\}, \tag{14}$$

for any time-varying delay $\tau(t) > 0$, namely, $e(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^{N} x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \zeta - 2\gamma\beta$, Furthermore, the NMAS (1) achieves bounded consensus for any fixed time delay $\tau(t) > 0$, $0 \leq \dot{\tau}(t) \leq 1$.

Proof. Choose the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N} V_i, \tag{15}$$

$$V_i = w_i^T P_i(t) w_i + \int_{t-\tau(t)}^{t} w_i^T(\alpha)Q_i w_i(\alpha)d\alpha. \tag{16}$$

Differentiating (16) along the trajectory of (8) gives

$$\dot{V}_i = w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i)w_i\\ + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w\\ + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) + 2w_i^T(c\lambda_i P_i(t)\Gamma)w_i(t-\tau(t))\\ - w_i^T(t-\tau(t))Q_i w_i(t-\tau(t)). \tag{17}$$

Applying the Young Inequality to the equality (17) results in

$$\dot{V}_i \leq w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i\\ + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t)\\ + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w. \tag{18}$$

Condition (10) implies that the first term on the right hand side of (18) satisfies

$$w_i^T(\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) + Q_i\\ + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t))w_i \leq -\zeta\|w_i\|^2. \tag{19}$$

The second term on the right hand side of (18) satisfies

$$2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t) \leq 2\mu(t)\|P_i(t)\|\|w_i\|. \tag{20}$$

Applying condition (11) we know the third term on the right hand side of (18) satisfies

$$2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \leq 2\gamma\|P_i(t)\|\|w_i\|\|w\|. \tag{21}$$

Since $V = \sum_{i=2}^{N} V_i$, we have

$$\begin{aligned}\dot{V} &= \sum_{i=2}^{N} \dot{V}_i\\ &= -\zeta\|w\|^2 + 2(\gamma\|w\| + \mu(t))\sum_{i=2}^{N}\|w_i\|\|P_i(t)\|\\ &\leq -\zeta\|w\|^2 + 2(\gamma\|w\| + \mu(t))\|w\|(\sum_{i=2}^{N}\|P_i(t)\|^2)^{\frac{1}{2}}\\ &= \|w\|((2\gamma\beta - \zeta)\|w\| + 2\beta\mu(t)).\end{aligned} \tag{22}$$

Thus, when

$$\|w\| \geq \frac{2\beta\mu(t)}{\zeta - 2\gamma\beta - \delta}, \tag{23}$$

we have

$$\dot{V} \leq -\delta\|w\|^2. \tag{24}$$

Applying the result in [22] completes the proof.

IV. Adaptive pinning controller

In this section, we will derive globally consensus criteria via direct adaptive pinning control method. Without loss of generality, we still assume that the first l agents are selected as pinned agents with the adaptive controllers:

$$\begin{cases} u_i = -d_i(x_i - s), & 1 \leq i \leq l,\\ \dot{d}_i = h_i e_i^T P_i(t) e_i, & \\ u_i = 0, & l+1 \leq i \leq N, \end{cases} \tag{25}$$

where constant $h_i > 0$ and positive definite matrix $P_i(t) \in R^{n\times n}$. Applying Newton-Leibniz formula, then the error

NMAS can be rewritten as

$$\begin{cases}\dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau(t)) \\ \qquad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \qquad -\frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \qquad + f_i(s) - \bar{f}(s) - d_i e_i, \qquad 1\le i\le l, \\ \dot{d}_i = h_i e_i^T P_i(t) e_i, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma e_j(t-\tau(t)) \\ \qquad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \qquad -\frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \qquad + f_i(s) - \bar{f}(s), \qquad l+1\le i\le N.\end{cases} \tag{26}$$

Repeating a similar procedure to the previous subsection, the controlled consensus problem of system (1) is equivalent to the stability problem of the following $N-1$ of n-dimensional systems.

$$\begin{cases}\dot{w}_i = D\bar{f}(s(t))w_i - d_i w_i + c\lambda_i\Gamma w_i(t-\tau(t)) \\ \qquad +(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w \\ \qquad +(\Phi_i^T\otimes I_n)F(t), \qquad 2\le i\le l, \\ \dot{d}_i = h_i w_i^T P_i(t) w_i, \\ \dot{w}_i = D\bar{f}(s)w_i + c\lambda_i\Gamma w_i(t-\tau(t)) \\ \qquad +(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w \\ \qquad +(\Phi_i^T\otimes I_n)F(t), \qquad l+1\le i\le N,\end{cases} \tag{27}$$

where w_i, w, Φ, Φ_i, $I(t)$ and $F(t)$ are the same as the previous subsection.

Theorem 2 Suppose there exist positive definite matrices $P_i(t)\in\mathcal{PC}^1_{n\times n}$, Q_i and constants $\bar{\zeta}>0$, $\gamma\ge 0$, $a>0$ and $b>0$ such that

$$a\|x\|^2 \le x_i^T P_i(t) x_i + \int_{t-\tau(t)}^t x_i^T(\alpha) Q_i x_i(\alpha) d\alpha \\ + \frac{(d_i-d)^2}{h_i} \le b\|x\|^2, \forall t\in R^+,\ x\in R^n, i=2,3,\cdots,N, \tag{28}$$

$$\dot{P}_i(t) + P_i(t)D\bar{f}(s) + (D\bar{f}(s))^T P_i(t) + Q_i - 2dP_i(t) \\ + c^2\lambda_i^2 P_i(t)\Gamma Q_i^{-1}\Gamma^T P_i(t) + \bar{\zeta} I \le 0,\ i=1,2,\cdots,N, \tag{29}$$

(11) and $\bar{\zeta} > 2\gamma\beta$ are satisfied, then the system (6) converges to the set (14) for any time-varying delay $\tau(t)>0$, where $\mu(t)$ and β are the same as in (12) and (13) respectively, $\bar{\delta}>0$ is any constant satisfying $\bar{\delta} < \bar{\zeta} - 2\gamma\beta$, and then the NMAS (1) achieves bounded consensus for any fixed time delay $\tau(t)>0$, $0\le\dot{\tau}(t)\le 1$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^N V_i + \sum_{i=2}^l \frac{(d_i-d)^2}{h_i}, \tag{30}$$

where

$$\begin{cases} V_i = w_i^T P_i(t) w_i + \int_{t-\tau(t)}^t w_i^T(\alpha) Q_i w_i(\alpha) d\alpha \\ \qquad + \frac{(d_i-d)^2}{h_i}, \qquad 2\le i\le l, \\ V_i = w_i^T P_i(t) w_i + \int_{t-\tau(t)}^t w_i^T(\alpha) Q_i w_i(\alpha) d\alpha, \\ \qquad\qquad l+1\le i\le N,\end{cases} \tag{31}$$

where d is a positive constant to be determined.

Differentiating (31) along the trajectory of (27) gives

$$\dot{V}_i = w_i^T(\dot{P}_i(t) + P_i(t)D\bar{f}(s) + (D\bar{f}(s))^T P_i(t) + Q_i \\ - 2dP_i(t))w_i + 2w_i^T P_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w \\ + 2w_i^T P_i(t)(\Phi_i^T\otimes I_n)F(t) + 2w_i^T(c\lambda_i P_i(t)\Gamma)w_i(t-\tau(t)) \\ - w_i^T(t-\tau(t))Q_i w_i(t-\tau(t)). \tag{32}$$

The remaining part of the proof is similar to that of Theorem 1, so is therefore omitted here. This completes the proof.

V. Examples

To demonstrate the theoretical results obtained above, we construct a NMAS consisting of 12 agents described as follows

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j\in\mathcal{N}_i} a_{ij}\Gamma x_j(t-\tau(t)), \tag{33}$$

where $f_i(x_i(t)) = B_i x_i(t) + g(x_i(t))$, $B_i(i=1,2,\cdots,6)$ and $B_i(i=7,8,\cdots,12)$ are chosen as follows:

$$\begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix},$$
$$\begin{pmatrix} -10-0.1\times(i-6) & 10+0.1\times(i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15+0.1\times(i-6) & 0 \end{pmatrix},$$

and

$$g(x_i(t)) = (-9.5sin(\frac{\pi x_{i1}(t)}{3.2}+\pi)\ 0\ 0)^T, \quad i=1,2,\cdots,12.$$

The communication coupling matrix $C=(C_1^T C_2^T\cdots C_{12}^T)$, $C_1=(-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0)$, $C_2=(1\ -8\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0)$, $C_3=(1\ 1\ -7\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1)$, $C_4=(0\ 1\ 1\ -6\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1)$, $C_5=(1\ 1\ 0\ 0\ -6\ 0\ 1\ 1\ 1\ 1\ 0\ 0)$, $C_6=(1\ 0\ 0\ 1\ 0\ -5\ 1\ 0\ 1\ 1\ 0)$, $C_7=(1\ 1\ 0\ 1\ 1\ 1\ -7\ 1\ 0\ 1\ 0)$, $C_8=(0\ 0\ 1\ 0\ 1\ 0\ 1\ -6\ 0\ 1\ 1\ 1)$, $C_9=(1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ -7\ 1\ 1\ 1)$, $C_{10}=(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ -10\ 1)$, $C_{11}=(1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ -7\ 1)$, $C_{12}=(0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ -5)$. $\Gamma = diag\{2,2,2\}$, respectively, where the matrix A is produced by means of the Scale-Free network program.

Design the following controllers

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k}(t)-s(t)), & i_k = 1,2 \text{ and } 10, \\ u_{i_k} = 0, & \text{else}, \end{cases}$$

with $d_1=0.5$, $d_2=0.5$, $d_{10}=0.5$ and

$$\begin{cases} u_{i_k} = -d_{i_k}(t)(x_{i_k}(t)-s(t)), & i_k = 1,2 \text{ and } 10, \\ \dot{d}_{i_k}(t) = h_{i_k} e_{i_k}^T P_{i_k}(t) e_{i_k}, \\ u_{i_k} = 0, & \text{else}, \end{cases}$$

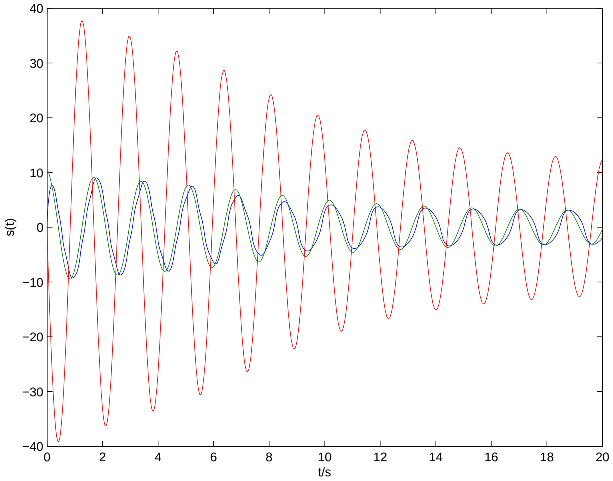

Fig.1. Desired agent dynamics under pinning control.

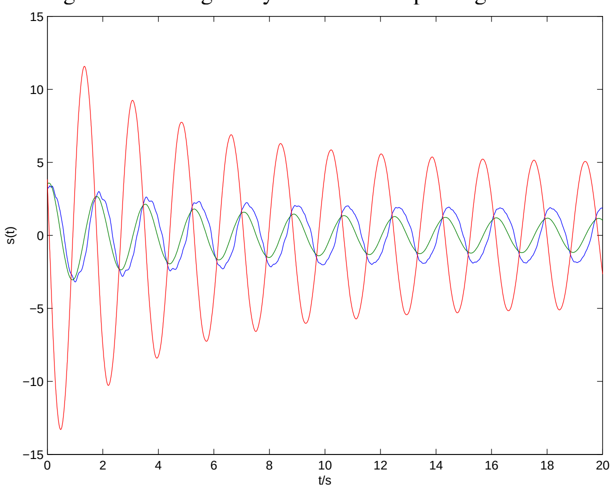

Fig.2. Desired agent dynamics under adaptive pinning control.

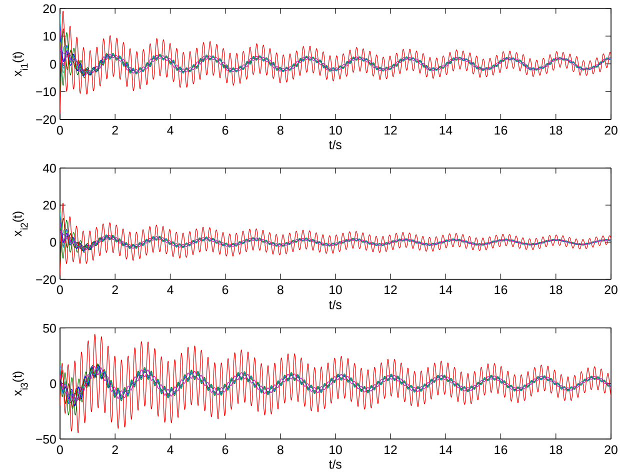

Fig.3. All agent dynamics under pinning control.

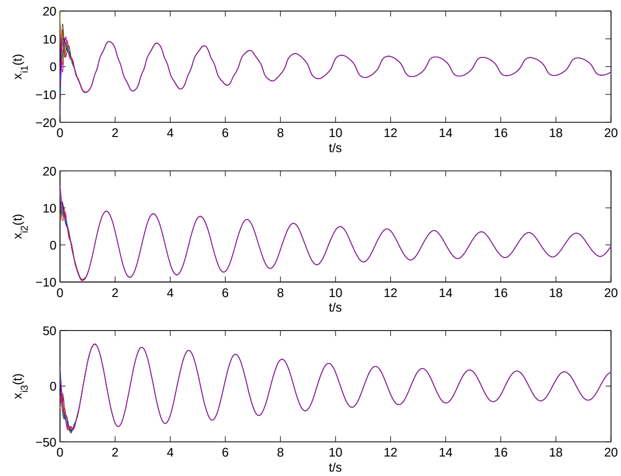

Fig.4. All agent dynamics under adaptive pinning control.

with $h_1 = 0.1$, $h_2 = 0.2$, $h_{10} = 0.3$, $s(t)$ can then be evaluated by simulation.

Given the initial values of 12 agents as $(10\ 5\ \ -10)^T$, $(12\ 6\ \ -12)^T$, $(14\ 7\ \ -14)^T$, $(16\ 8\ \ -16)^T$, $(18\ 9\ \ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$, $(-10\ 15\ 10)^T$, $(-8\ 16\ 8)^T$ respectively and $P_{i_k}(t) = I_3$, $d_1(0) = 1$, $d_2(0) = 1$, $d_{10}(0) = 1$ and $\tau(t) = \frac{\pi}{2} + arctan(t)$. The conditions of Theorem 1 and Theorem 2 are satisfied readily. Bounded consensus of the NMAS is achieved for any time varying delay satisfying $0 < \tau \leq \frac{\pi}{2} + arctan(t)$. Simulation results are depicted in Fig.1 to Fig.8 for $\tau(t) = \frac{\pi}{2} + arctan(t)$ and $c = 1$.

VI. Conclusion

In this paper, we've investigated the controlled consensus problems of NMAS with different agent dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on pinning control and adaptive pinning control methods. Many related results for the case of identical agent dynamics have been viewed as the special cases of the proposed results. However, it should be noted that the conditions are still restrictive and the time-varying delay is chosen as fixed case. Further investigations will focus on relaxing these limitations and more generalized cases.

References

[1] Sandro Zampieri. Trends in networked control systems. *Proceedings of 17th World Congress of IFAC*, Seoul, Korea, July 6-11, 2008, pp. 2886 – 2894.

[2] J.P. Desai, J.P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust. *IEEE Int. Trans. on Robotics and Automation*, 2001, 17(6): 905 – 908.

[3] M. Porfiri, D.G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach. *Automatica*, 2007, 43: 1318 – 1328.

[4] J. Cortés. Global formation-shape stabilization of relative sensing networks. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 1460 – 1465.

[5] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Transactions on Automatic Control*, 2008, 53(8): 1804 – 1816.

[6] T. Li and J.F. Zhang. Decentralized tracking-type games for multi-agent systems with coupled ARX models: asymptotic Nash equilibria. *Automatica*, 2008, 44: 713 – 725.

[7] E.S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information. *Automatica*, 2008, 44: 2766 – 2777.

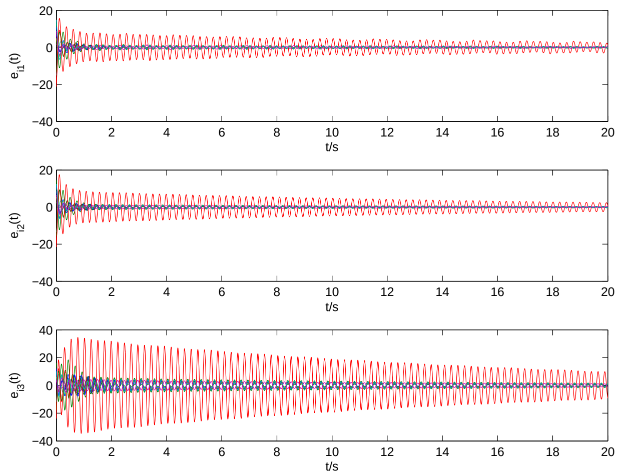

Fig.5. All agent error dynamics under pinning control.

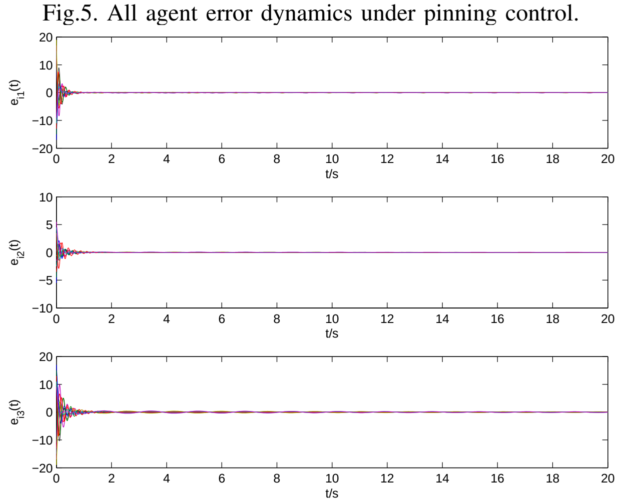

Fig.6. All agent error dynamics under adaptive pinning control.

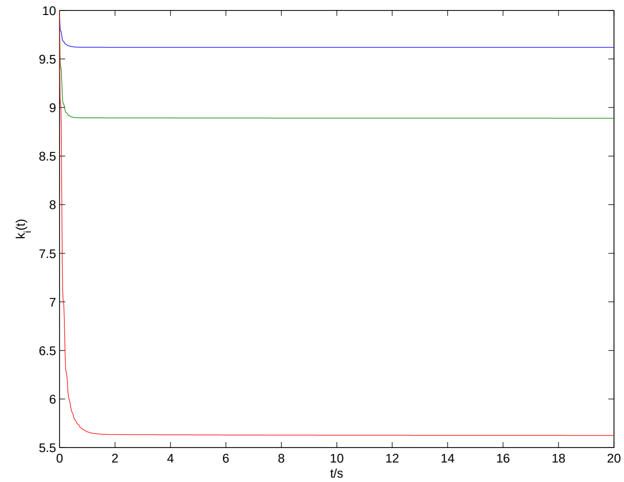

Fig.7. Adaptive gain curves.

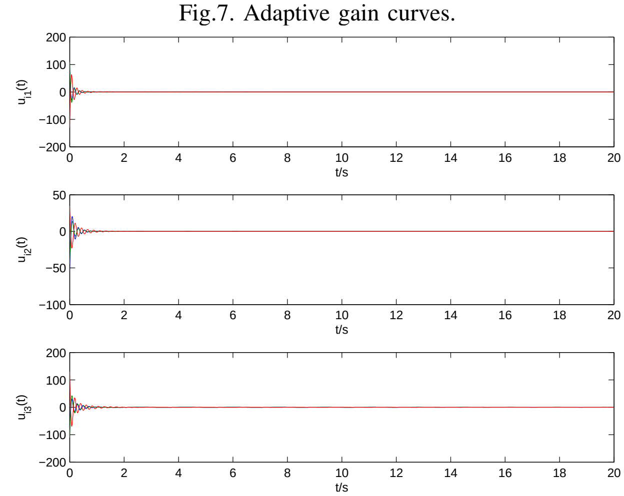

Fig.8. Adaptive pinning controllers curves.

[8] Z.K. Li, Z.S. Duan, G.R. Chen and L. Huang. Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint. *IEEE Transactions on Circuits and Systems-I*, 2010, 57(1): 213 – 224.

[9] R. Olfati-Saber and R.M. Murray. Consensus problmes in networks with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1520 – 1533.

[10] M. Arack. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 2007, 52(8): 1380 – 1390.

[11] F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu and Z.Z. Yuan. Reaching a consensus via pinning control. *Automatica*, 2009, 45: 1215 – 1220.

[12] Y.P. Tian and C.L Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations. *Automatica*, 2009, 45: 1347 – 1353.

[13] P. Lin, Y.M. Jia and L. Li. Distributed robust H_∞ Consensus control in directed networks of agents with time-delay. *Systems and Control Letters*, 2008, 57(8): 643 – 653.

[14] W. Ren, R.W. Beard. Formation feedback control for multiple spacecraft via virtual structures. *IEE Proc. Control Theory and Applications*, 2004, 151(3): 357 – 368.

[15] H. Yamaguchi, T. Arai and G. Beni. A distributed control scheme for multiple robotic vehicles to make group formations. *Robitics and Autonomous Systems*, 2001, 36: 125-147.

[16] Z.P. Wu, Z.H. Guan, T. Li. Group motion control of multi-agent systems based on complex network, *Proceedings of the 26th Chinese Control Conference*, Hunan, China, July 26-31, 2007, pp: 63 – 67.

[17] J.Q. Lu, D.W.C. Ho and J. Kurths. Consensus over directed static networks with arbitrary finite communication delays. *Physical Review E*, 2009, 80(6): 066121.

[18] J. Cortes. Distributed algorithms for reaching consensus on general functions. *Automatica*, 2008, 44(3): 726 – 737.

[19] J. Zhao, D. J. Hill and T. Liu. Synchronization of Dynamical Networks With Nonidentical Nodes: Criteria and Control. *IEEE Tran.Circuits and Syst.I*, 2011, 58(3): 584 – 594.

[20] P.A. Bliman and G.F. Trecate. Average consensus problems in networks of agents with delayed communications. *Automatica*, 2008, 44: 1985 – 1995.

[21] Z.S Duan and G.R. Chen. Global Robust Stability and Synchronization of Networks With Lorenz-Type Nodes. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, 2009, 56(8): 679 – 683.

[22] D. J.Hill and J. Zhao. Global synchronization of complex dynamical networks with non-identical nodes. *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 9-11, 2008, 817 – 822.

[23] J.C. Zhao, J.A. Lu and Q.J. Zhang. Pinning a complex dynamical network to a homogenous trajectory. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, 2009, 56(6): 514 – 518.

[24] X. Li, X.F. Wang and G.R. Chen. Pinning a complex dynamical network to its equilibrium. *IEEE Tran.Circuits and Syst.I*,2004, 51: 2074 – 2087.

[25] W. Wu, W.J. Zhou and T.P. Chen. Cluster Synchronization of Linearly Coupled Complex Networks Under Pinning Control. *IEEE Transactions on Circuits and Systems-I*, 2009, 56(4): 829 – 839.

[26] Y.Y. Wu, W. Wei, G.Y. Li and J. Xiang. Pinning Control of Uncertain Complex Networks to a Homogeneous Orbit. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, 2009, 56(3): 235 – 239.

:

Global Bounded Controlled Consensus of Multi-Agents Systems with Non-Identical Nodes and Communication Time-Delay Topology

Wei-Song Zhong
Faculty of Advanced Technology
University of Glamorgan
Cardiff, UK CF37 1DL
Email: wzhong@glam.ac.uk

Guo-Ping Liu
Faculty of Advanced Technology
University of Glamorgan
Cardiff, UK CF37 1DL
Email: gpliu@glam.ac.uk

Clive Thomas
Faculty of Advanced Technology
University of Glamorgan
Cardiff, UK CF37 1DL
Email: ejcthomas@glam.ac.uk

***Abstract*—This paper investigates the global bounded consensus problem of Networked Multi-Agent Systems exhibiting nonlinear, non-identical agent dynamics with communication time-varying delay. Globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived. The proposed consensus criteria ensures that all agents eventually move along desired trajectories in terms of boundedness. The proposed controlled consensus criteria generalizes the case of identical agent dynamics to the case of non-identical agent dynamics, and many related results of other researches in this area can be viewed as special cases of the above results. We finally demonstrate the effectiveness of the theoretical results by means of a numerical simulation.**

I. Introduction

Networked Multi-Agent Systems (NMAS) analysis involves the study of how the network architectures and interactions between network components influence global control goals and some important contributions have been made in recent years [1], [2], [3], [4].

The consensus problem has been studied across many fields of science and engineering [5], [6], [7], [8], [9], [10], [11], [12], [13]. The controlled consensus problem of NMAS with non-identical agent dynamics is much more complicated than the identical case and few results have been reported to date [14].

The present paper will focus on the global consensus problems of NMAS based on pinning control methods [15], [16], [17], and the proposed controlled consensus property is formulated in terms of certain boundedness of state errors. In this paper, we'll generalize many existing results for the case of identical agent dynamics to the case of non-identical agent dynamics based on the pinning control method.

The rest of this paper is organized as follows. A controlled continuous-time NMAS model with communication time-delay is presented in Section II. The main results including pinning control and adaptive pinning control bounded consensus criterion are derived in Section III and V respectively. Section IV gives a numerical simulation example to verify the effectiveness of the proposed results, followed by conclusions in Section VI.

II. Problem description

Let $G = (\mathcal{V}, \mathcal{A})$ be a graph of order N consisting of a set of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and a set of edges $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$. An edge (v_j, v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{A}\}$.

We consider a MAS consisting of N non-identical agents with communication delay:

$$\dot{x}_i = f_i(x_i) + c \sum_{j \in \mathcal{N}_i}^{N} a_{ij} \Gamma x_j(t - \tau), i = 1, 2, \cdots, N, \tag{1}$$

where $x_i = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n \times n}$ is the inner coupling matrix, and where $\gamma_{ij} \neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N \times N}$ (which is symmetric and irreducible) represents the communication topology relation of the MAS, and is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j \neq i} a_{ij}$. τ is a constant coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field $\bar{f}(x(t)) = \frac{1}{N} \sum_{k=1}^{N} f_k(x(t))$ with Jacobian $D\bar{f}_i(x(t))$.

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N} \sum_{k=1}^{N} x_k(t). \tag{2}$$

We now discuss the problem of global consensus for the system (1). The consensus problem formulation in the present paper is quite different from many others, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)$, $\forall i, j = 1, 2, \cdots, N$ as $t \to \infty$. The consensus problem here will be depicted instead via certain boundedness of $x_i(t)-x_j(t)$, $\forall i,j=1,2,\cdots,N$ as $t\to\infty$. This better reflects reality as it is impossible for MAS (1) to achieve exact consensus. To address this case we will focus on making the states of all agents converge to a bounded set.

We denote $x(t)$, $s(t)$, $u(t)$, $e(t)$, $w(t)$ and $V(w(t),t)$ as x, s, u, e, w and V respectively.

III. LINEAR FEEDBACK PINNING CONTROLLER

To achieve the goal, we apply the feedback control strategy on a small fraction δ $(0<\delta\leq 1)$ of the agents in system (1). Suppose that nodes $i_1,i_2,\cdots,i_l$ are selected to be under control, where $l=[\delta N]$ stands for the smaller but nearest integer to the real number δN. This controlled MAS can be described as

$$\begin{cases}\dot{x}_{i_k}=f_{i_k}(x_{i_k})+c\sum_{j=1}^{N}a_{i_kj}\Gamma x_j(t-\tau)+u_{i_k}, 1\leq k\leq l,\\ \dot{x}_{i_k}=f_{i_k}(x_{i_k})+c\sum_{j=1}^{N}a_{i_kj}\Gamma x_j(t-\tau), l+1\leq k\leq N.\end{cases} \tag{3}$$

The local linear negative feedback control law is chosen as follows:

$$\begin{cases}u_{i_k}=-d_{i_k}(x_{i_k}-s), & 1\leq k\leq l,\\ u_{i_k}=0, & l+1\leq k\leq N,\end{cases} \tag{4}$$

where the feedback gain $d_{i_k}>0$.

Combine (3) and (4) and rearrange the order of the nodes in the network. Let the first l nodes be controlled, and $e_i=x_i-s$, $i=1,2,\cdots,N$. It's obvious that $\frac{c}{N}\sum_{k=1}^{N}\sum_{j=1}^{N}a_{kj}\Gamma x_j(t-\tau)=0$ and $\sum_{i=1}^{N}e_i=0$. Then by applying the Newton-Leibniz formula, error systems can be written as

$$\begin{cases}\dot{e}_i=D\bar{f}(s)e_i+c\sum_{j=1}^{N}a_{ij}\Gamma e_j(t-\tau)\\ \quad+\int_0^1(Df_i(s+\tau e_i)-D\bar{f}(s))e_id\tau\\ \quad-\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_kd\tau\\ \quad+f_i(s)-\bar{f}(s)-d_ie_i, & 1\leq i\leq l,\\ \dot{e}_i=D\bar{f}(s)e_i+c\sum_{j=1}^{N}a_{ij}\Gamma e_j(t-\tau)\\ \quad+\int_0^1(Df_i+\tau e_i)-D\bar{f}(s))e_id\tau\\ \quad-\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_kd\tau\\ \quad+f_i(s)-\bar{f}(s), & l+1\leq i\leq N.\end{cases} \tag{5}$$

The following work will focus on simplifying the error systems (5) by means of a series of transformations using a procedure similar to [14].

Define the following matrix

$$D=diag(D_1,D_2,\cdots,D_N)\in R^{nN\times nN},$$

where $D_i=diag\{-d_i,-d_i,\cdots,-d_i\}\in R^{n\times n}$.

Let $e=(e_1^T,e_2^T,\cdots,e_N^T)^T$, then (5) becomes

$$\dot{e}=\bar{\Sigma}(t)e+cA\otimes\Gamma e(t-\tau)+I(t)e-\frac{1}{N}H(t)e+F(t), \tag{6}$$

where $I(t)=diag\{\int_0^1(Df_1(s+\tau e_1)-D\bar{f}(s))d\tau\cdots\int_0^1(Df_N(s+\tau e_N)-D\bar{f}(s))d\tau\}$, $\bar{\Sigma}(t)=I_N\otimes D\bar{f}(s)+D$, $H^T(t)=(H_1^T(t),\cdots,H_N^T(t))$, $H_i(t)=(\int_0^1 Df_1(s+\tau e_1)d\tau,\cdots,\int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t)=(f_1^T(s)-\bar{f}^T(s),\cdots,f_N^T(s)-\bar{f}^T(s))$.

Since A is symmetric and irreducible, according to [14], there exists a unitary matrix $\Phi=(\varphi_{ij})_{N\times N}=(\Phi_1,\Phi_2,\cdots,\Phi_N)$. This together with $w(t)=(\Phi^T\otimes I_n)e$ gives

$$\begin{aligned}\dot{w}&=(\Phi^T\otimes I_n)\bar{\Sigma}(t)(\Phi\otimes I_n)w\\&+(\Phi^T\otimes I_n)(cA\otimes\Gamma)(\Phi\otimes I_n)w(t-\tau)\\&+(\Phi^T\otimes I_n)I(t)(\Phi\otimes I_n)w\\&-\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)w+(\Phi^T\otimes I_n)F(t).\end{aligned} \tag{7}$$

Note that $H(t)=\sqrt{N}\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0})\otimes\int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its k-th column equal to Φ_1 and the remaining elements are zero. Then we have $\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)=\frac{1}{\sqrt{N}}\sum_{k=1}^{N}(\mathbf{0}\ \cdots\ \mathbf{0}\ I_k\ \mathbf{0}\ \cdots\ \mathbf{0})\otimes\int_0^1 Df_k(s+\tau e_k)d\tau(\Phi\otimes I_n)$, where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the remaining of its elements are zero.

Thus, a simple calculation gives $\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)=\frac{1}{\sqrt{N}}\sum_{k=1}^{N}\begin{pmatrix}\Upsilon_k\\ 0\end{pmatrix}\otimes\int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k\in R^{1\times N}$ and $0\in R^{(N-1)\times N}$. Therefore, $\dot{w}=\bar{\Sigma}(t)w+c\Lambda\otimes\Gamma w(t-\tau)+(\Phi^T\otimes I_n)I(t)(\Phi\otimes I_n)w-\begin{pmatrix}*\\ 0\end{pmatrix}w+(\Phi^T\otimes I_n)F(t)$. Since $w_1\equiv 0$, we only need to consider $w_2,w_3,\cdots,w_N$. Rewriting in the component form we have

$$\begin{aligned}\dot{w}_i&=\Sigma_i(t)w_i+c\lambda_i\Gamma w_i(t-\tau)+(\Phi_i^T\otimes I_n)F(t)\\&+(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w,\ i=2,3,\cdots,N,\end{aligned} \tag{8}$$

where $\Sigma_i=\bar{D}f(s)+D_i$.

So far, we have transferred the consensus problem of system (1) to the stability problem of the $N-1$ of $n-$dimensional systems.

Theorem 1 Suppose that $\|I(t)\|\leq\gamma$ is satisfied. If there exist matrices $P_i(t)\in\mathcal{PC}^1_{n\times n}$, $Q_i>0$, $\Theta_i>0$, $\Pi_i>0$, X_i, Y_i and Z_i of appropriate dimensions such that

$$B=\begin{pmatrix}B_1 & B_2\\ B_2^T & B_3\end{pmatrix}<0,\ \begin{pmatrix}X_i & Y_i\\ Y_i^T & Z_i\end{pmatrix}\geq 0, \tag{9}$$

for $i=2,3,\cdots,N$, where $B_1=\dot{P}_i(t)+P_i(t)\Sigma_i(t)+\Sigma_i^T(t)P_i(t)+hX_i+Y_i^T+Y_i+Q_i+h\Sigma_i^T(t)Z_i\Sigma_i(t)$, $B_2=c\lambda_iP_i(t)\Gamma-Y_i+hc\lambda_i\Sigma_i^T(t)Z_i\Gamma$ and $B_3=\Pi_i^{-1}+\Theta_i^{-1}-Q_i+hc^2\lambda_i^2\Gamma^TZ_i\Gamma$, then the MAS (1) will achieve bounded consensus for the time-invariant delay $\tau\in[0,h]$ for some $h<\infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V=\sum_{i=2}^{N}\sum_{k=1}^{3}V_k, \tag{10}$$

where

$$V_1 = w_i^T P_i(t) w_i,$$
$$V_2 = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha d\beta,$$
$$V_3 = \int_{t-\tau}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha.$$

The i-th $(i = 2, 3, \cdots, N)$ equation in system (8) can be written as

$$\dot{w}_i = (\Sigma_i(t) + c\lambda_i\Gamma)w_i - c\lambda_i\Gamma\int_{t-\tau}^{t}\dot{w}_i(\alpha)d\alpha + (\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w + (\Phi_i^T \otimes I_n)F(t). \quad (11)$$

Defining $a(.)$, $b(.)$ and M in [18] as $a(\alpha) = w_i(t)$, $b(\alpha) = \dot{w}_i(\alpha)$ and $M = c\lambda_i P_i(t)\Gamma$ for all $\alpha \in [t-\tau, t]$ then we have

$$\begin{aligned}\dot{V}_1 \le\ & w_i^T[\dot{P}_i(t) + P_i(t)\Sigma_i(t) + \Sigma_i^T(t)P_i(t) \\ & + hX_i + Y_i^T + Y_i]w_i + \int_{t-\tau}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha \\ & + 2w_i^T(c\lambda_i P_i(t)\Gamma - Y_i)w_i(t-\tau) \\ & + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)I(t)(\Phi_i \otimes I_n)w \\ & + 2w_i^T P_i(t)(\Phi_i^T \otimes I_n)F(t). \qquad (12)\end{aligned}$$

Moreover, $\dot{V}_2$ can be enlarged as

$$\begin{aligned}\dot{V}_2 \le\ & h[\Sigma_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau)]^T Z_i[\Sigma_i(t)w_i \\ & + c\lambda_i\Gamma w_i(t-\tau)] + 2h(\Sigma_i(t)w_i)^T Z_i(\Phi_i^T \otimes I_n)I(t) \\ & (\Phi \otimes I_n)w + 2h(\Sigma_i(t)w_i)^T Z_i(\Phi_i^T \otimes I_n)F(t) \\ & + 2h(c\lambda_i\Gamma w_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ & + 2h(c\lambda_i\Gamma w_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)F(t) \\ & + 2h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w)^T Z_i(\Phi_i^T \otimes I_n)F(t) \\ & + h((\Phi_i^T \otimes I_n)F(t))^T Z_i((\Phi_i^T \otimes I_n)F(t)) \\ & + h((\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w)^T Z_i((\Phi_i^T \otimes I_n)I(t) \\ & (\Phi \otimes I_n)w) - \int_{t-\tau}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha. \qquad (13)\end{aligned}$$

and

$$\dot{V}_3 = w_i^T Q_i w_i - w_i^T(t-\tau)Q_i w_i(t-\tau). \quad (14)$$

Applying the Young Inequality, then we have $2h(c\lambda_i\Gamma w_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \le w_i^T(t-\tau)\Pi_i^{-1}w_i(t-\tau) + h^2c^2\lambda_i^2 w^T((\Phi \otimes I_n)^T I(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Pi_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n))w(t)$, and $2h(c\lambda_i\Gamma w_i(t-\tau))^T Z_i(\Phi_i^T \otimes I_n)F(t) \le w_i^T(t-\tau)\Theta_i^{-1}w_i(t-\tau) + h^2c^2\lambda_i^2 F^T(t)(\Phi_i^T \otimes I_n)^T Z_i\Gamma\Theta_i\Gamma^T Z_i(\Phi_i^T \otimes I_n)F(t)$. Applying these two inequalities and the conditions of the theorem results

$$\begin{aligned}\dot{V} \le\ & \sum_{i=2}^{N}\begin{pmatrix} w_i \\ w_i(t-\tau)\end{pmatrix}^T B \begin{pmatrix} w_i \\ w_i(t-\tau)\end{pmatrix} \\ & + 2\mu(t)\beta + (\|w\|(2\gamma\beta + 2h\gamma\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i) \\ & + 2h\mu(t)\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + h\gamma^2\sum_{i=2}^{N}\lambda_{max}(Z_i) \\ & + h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) \\ & + h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Theta_i)\lambda_i^2\lambda_{max}^2(Z_i))\|w\| \\ & + 2h\gamma\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)) + h\mu^2(t)\sum_{i=2}^{N}\lambda_i^2\lambda_{max}(Z_i). \qquad (15)\end{aligned}$$

Thus when

$$\|w\| \ge \frac{2\mu(t)\beta + 2h\gamma\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)}{\varpi(t)},$$

we have

$$\dot{V} \le -\delta\|w\|^2 + h\mu^2(t)\sum_{i=2}^{N}\lambda_{max}(Z_i)\lambda_i^2, \quad (16)$$

where $\varpi(t) = -(2\gamma\beta + 2h\gamma\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + 2h\mu(t)\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i) + h\gamma^2\sum_{i=2}^{N}\lambda_{max}(Z_i) + h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i) + h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Theta_i)\lambda_i^2\lambda_{max}^2(Z_i)) - \delta$. Thus, according to [19] and Lyapunov stability theory, bounded consensus is ultimately achieved. This completes the proof.

IV. Adaptive pinning controller

In this section, we will derive globally consensus criteria via direct adaptive pinning control method. Without loss of generality, we still assume that the first l agents are selected as pinned agents with the adaptive controllers:

$$\begin{cases} u_i = -d_i(t)(x_i - s), & 1 \le i \le l, \\ \dot{d}_i(t) = h_i e_i^T P_i(t) e_i, & \\ u_i = 0, & l+1 \le i \le N, \end{cases} \quad (17)$$

where constant $h_i > 0$ and positive definite matrix $P_i(t) \in R^{n\times n}$. Applying Newton-Leibniz formula, then the error MAS can be rewritten as

$$\begin{cases} \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j=1}^N a_{ij}\Gamma e_j(t-\tau) \\ \qquad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \qquad -\frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \qquad +f_i(s) - \bar{f}(s) - d_i(t)e_i, \qquad 1 \le i \le l, \\ \dot{d}_i(t) = h_i e_i^T P_i(t) e_i, \\ \dot{e}_i = D\bar{f}(s)e_i + c\sum_{j=1}^N a_{ij}\Gamma e_j(t-\tau) \\ \qquad + \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\ \qquad -\frac{1}{N}\sum_{k=1}^N \int_0^1 Df_k(s+\tau e_k)e_k d\tau \\ \qquad +f_i(s) - \bar{f}(s), \qquad l+1 \le i \le N. \end{cases} \tag{18}$$

Repeating a similar procedure to the previous subsection, the controlled consensus problem of system (1) is equivalent to the stability problem of the following $N-1$ of n-dimensional systems.

$$\begin{cases} \dot{w}_i = D\bar{f}(s(t))w_i - d_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau) \\ \qquad +(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \qquad +(\Phi_i^T \otimes I_n)F(t), \qquad 2 \le i \le l, \\ \dot{d}_i(t) = h_i w_i^T P_i(t) w_i, \\ \dot{w}_i = D\bar{f}(s)w_i + c\lambda_i\Gamma w_i(t-\tau) \\ \qquad +(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \qquad +(\Phi_i^T \otimes I_n)F(t), \qquad l+1 \le i \le N, \end{cases} \tag{19}$$

where w_i, w, Φ, Φ_i, $I(t)$ and $F(t)$ are the same as the previous subsection.

Theorem 2 Suppose that $\|I(t)\| \le \gamma$ is satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Theta_i > 0$, $\Pi_i > 0$, X_i, Y_i and Z_i of appropriate dimensions and constant $d > 0$ such that

$$B = \begin{pmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{pmatrix} < 0, \quad \begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \ge 0, \tag{20}$$

for $i = 2, 3, \cdots, N$, where $B_1 = \dot{P}_i(t) + P_i(t)(Df(s)) + (Df(s))^T P_i(t) - 2dP_i(t) + hX_i + Y_i^T + Y_i + Q_i + h\Sigma_i^T(t)Z_i\Sigma_i(t)$, $B_2 = c\lambda_i P_i(t)\Gamma - Y_i + hc\lambda_i\Sigma_i^T(t)Z_i\Gamma$ and $B_3 = \Pi_i^{-1} + \Theta_i^{-1} - Q_i + hc^2\lambda_i^2\Gamma^T Z_i\Gamma$, then the system (1) will achieve bounded consensus for the time-invariant delay $\tau \in [0, h]$ for some $h < \infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^N \sum_{k=1}^3 V_k + \sum_{i=2}^l \frac{(d_i(t)-d)^2}{h_i}, \tag{21}$$

where

$$V_1 = w_i^T P_i(t) w_i,$$
$$V_2 = \int_{-\tau}^0 \int_{t+\beta}^t \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha d\beta,$$
$$V_3 = \int_{t-\tau}^t w_i^T(\alpha) Q_i w_i(\alpha) d\alpha.$$

The remaining part of the proof is similar to that of Theorem 1 and is therefore omitted here. This completes the proof.

V. Example

To demonstrate the theoretical results obtained above, we construct a MAS consisting of 11 agents described as follows

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j\in\mathcal{N}_i}^N a_{ij}\Gamma x_j(t-\tau), \tag{22}$$

where $f_i(x_i(t)) = B_i x_i(t) + g(x_i(t))$, $B_i(i = 1, 2, \cdots, 6)$ and $B_i(i = 7, 8, \cdots, 11)$ are chosen as follows:

$$\begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix},$$
$$\begin{pmatrix} -10-0.1\times(i-6) & 10+0.1\times(i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15+0.1\times(i-6) & 0 \end{pmatrix},$$

and

$$g(x_i(t)) = (-9.5sin(\frac{\pi x_{i1}(t)}{3.2} + \pi)\ 0\ 0)^T, \quad i = 1, 2, \cdots, 11.$$

The communication coupling matrix $C = (C_1^T C_2^T \cdots C_{11}^T)$, $C_1 = (-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1)$, $C_2 = (1\ -8\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1)$, $C_3 = (1\ 1\ -6\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1)$, $C_4 = (0\ 1\ 1\ -5\ 0\ 1\ 1\ 0\ 0\ 1\ 0)$, $C_5 = (1\ 1\ 0\ 0\ -6\ 0\ 1\ 1\ 1\ 1\ 0)$, $C_6 = (1\ 0\ 0\ 1\ 0\ -5\ 1\ 0\ 1\ 1\ 0)$, $C_7 = (1\ 1\ 0\ 1\ 1\ 1\ -7\ 1\ 0\ 1\ 0)$, $C_8 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ -5\ 0\ 1\ 1)$, $C_9 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ -6\ 1\ 1)$, $C_{10} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ -10\ 1)$, $C_{11} = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ -6)$. $\Gamma = diag\{2, 2, 2\}$, respectively, where the matrix A is produced by means of the Scale-Free network program.

Design the following controllers

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k}(t) - s(t)), & i_k = 1, 2 \text{ and } 10, \\ u_{i_k} = 0, & \text{else}, \end{cases}$$

with $d_1 = 0.5$, $d_2 = 0.5$, $d_{10} = 0.5$ and

$$\begin{cases} u_{i_k} = -d_{i_k}(t)(x_{i_k}(t) - s(t)), & i_k = 1, 2 \text{ and } 10, \\ \dot{d}_{i_k}(t) = h_{i_k} e_{i_k}^T P_{i_k}(t) e_{i_k}, \\ u_{i_k} = 0, & \text{else}, \end{cases}$$

with $h_1 = 0.1$, $h_2 = 0.2$, $h_{10} = 0.3$, $s(t)$ can then be evaluated by simulation.

Given the initial values of 11 agents as $(10\ 5\ -10)^T$, $(12\ 6\ -12)^T$, $(14\ 7\ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$, $(-10\ 15\ 10)^T$ respectively and $P_{i_k}(t) = I_3$. We may verify the conditions of Theorem 1 and Theorem 2 readily. This demonstrates the bounded consensus of the MAS is achieved for any time delay $0 < \tau \le 0.061$. Simulation results are depicted in Fig.1 to Fig.4 for $\tau = 0.061$ and $c = 1$.

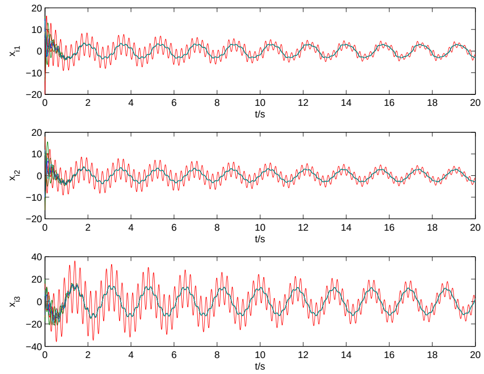

Fig.1. All agent dynamics under pinning control.

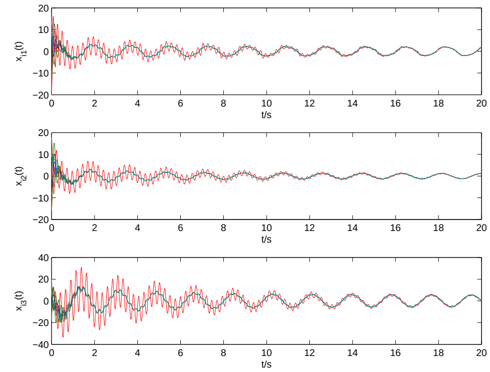

Fig.2. All agent dynamics under adaptive pinning control.

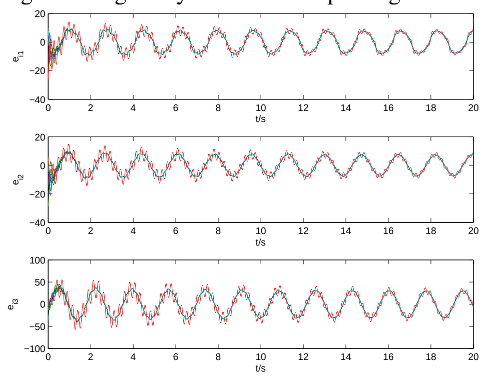

Fig.3. All agent error dynamics under pinning control.

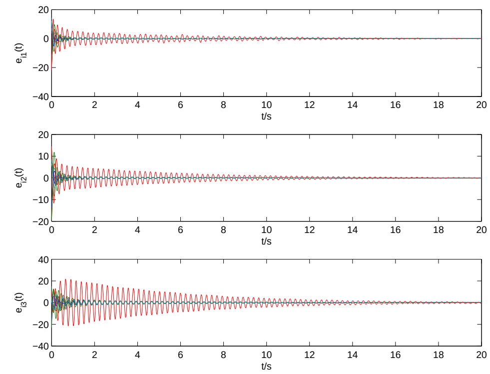

Fig.4. All agent error dynamics under adaptive pinning control.

VI. Conclusion

In this paper, we've investigated the controlled consensus problems of NMAS with different agent dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the NMAS is achieved based on pinning control and adaptive pinning control methods. It should be noted that the conditions are still restrictive and all the delays are the same. Further investigations will focus on relaxing these limitations.

References

[1] Sandro Zampieri. Trends in networked control systems, *Proceedings of 17th World Congress of IFAC*, Seoul, Korea, July 6-11, 2008, pp. 2886 – 2894.

[2] J.P. Desai, J.P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust. *IEEE Int. Trans. on Robotics and Automation*, 2001, 17(6): 905 – 908.

[3] M. Porfiri, D.G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach. *Automatica*, 2007, 43: 1318 – 1328.

[4] J. Cortés. Global formation-shape stabilization of relative sensing networks. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 1460 – 1465.

[5] Y.G. Hong, J.P. Hu and L.X. Gao. Tracking control for multi-agent consensus with an active leader and variable topology. *Automatica*, 2006, 42: 1177 – 1182.

[6] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays. *IEEE Transactions on Automatic Control*, 2008, 53(8): 1804 – 1816.

[7] E.S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information. *Automatica*, 2008, 44: 2766 – 2777.

[8] R. Olfati-Saber and R.M. Murray. Consensus problmes in networks with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1520 – 1533.

[9] M. Arack. Passivity as a design tool for group coordination. *IEEE Transactions on Automatic Control*, 2007, 52(8): 1380 – 1390.

[10] F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu and Z.Z. Yuan. Reaching a consensus via pinning control. *Automatica*, 2009, 45: 1215 – 1220.

[11] Y.P. Tian and C.L Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations. *Automatica*, 2009, 45: 1347 – 1353.

[12] P. Lin, Y.M. Jia and L. Li. Distributed robust H_∞ Consensus control in directed networks of agents with time-delay. *Systems and Control Letters*, 2008, 57(8): 643 – 653.

[13] P.A. Bliman and G.F. Trecate. Average consensus problems in networks of agents with delayed communications. *Automatica*, 2008, 44: 1985 – 1995.

[14] D. J.Hill and J. Zhao. Global synchronization of complex dynamical networks with non-identical nodes, *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 9-11, 2008, 817 – 822.

[15] J.C. Zhao, J.A. Lu and Q.J. Zhang. Pinning a complex dynamical network to a homogenous trajectory. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, 2009, 56(6): 514 – 518.

[16] W. Wu, W.J. Zhou and T.P. Chen. Cluster Synchronization of Linearly Coupled Complex Networks Under Pinning Control. *IEEE Transactions on Circuits and Systems-I*, 2009, 56(4): 829 – 839.

[17] Y.Y. Wu, W. Wei, G.Y. Li and J. Xiang. Pinning Control of Uncertain Complex Networks to a Homogeneous Orbit. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, 2009, 56(3): 235 – 239.

[18] Y.M. Moon, P. Park, W.H. Kwon, Y.S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems. *Int J. Control*, 2001, 74: 1447 – 1455.

[19] C.C. Hua, X.P. Guan and P. Shi. Decentralized robust model reference adaptive cotnrol for interconnected time-delay systems. *J.Comput.Appl.Math.*, 2006, 193: 383 – 396.

:

Global Bounded Controlled Consensus of Multi-Agents Systems with Non-Identical Nodes and Communication Time-Delay Topology

Wei-Song Zhong, Guo-Ping Liu, *Senior Member, IEEE,* David Rees and Clive Thomas

***Abstract*—This paper investigates the global bounded consensus problem of networked Multi-Agent Systems (MAS) exhibiting nonlinear, non-identical node dynamics with communication time-delays. Globally bounded controlled consensus conditions based on pinning control method and adaptive pinning control method are derived. The proposed consensus criteria ensures that all agents eventually move along desired trajectories. The proposed controlled consensus criteria generalizes the case of identical agent dynamics to the case of non-identical agent dynamics, and many related results of other researches in this area can be viewed as special cases of the above results. We finally demonstrate the effectiveness of the theoretical results by means of a numerical simulation.**

***Index Terms*—Networked control systems, multi-agent systems, consensus, pinning control, adaptive pinning control.**

I. Introduction

MULTI-Agent Systems (MAS) analysis involves the study of how the network architectures and interactions between network components influence global control goals. The research in this field can be categorized into two areas. One area is the design of distributed estimation techniques which can be applied to the sensor networks, and the other area is with the control of mobile autonomous agents i.e., each agent acts autonomously using information obtained over the network from other agents [1]. In both areas some important contributions have been made in recent years [2], [3], [4].

The consensus problem requires an agreement to be reached that depends on the state of all agents. The topic has been studied across many fields of science and engineering [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]. It is noted that the agent dynamics in most existing works are often restricted to linear and identical ones. Obviously, in practice, this is not always the case. The controlled consensus problem of MAS with non-identical agent dynamics is much more complicated than the identical case and few results have been reported to date [16], [17].

The present paper will focus on the global consensus problems of MAS based on pinning control methods [18], [19], [20], and the proposed controlled consensus property is formulated in terms of certain boundedness of state errors. Compared with existing related results, this paper make two significant advances. One is that we generalize the related results for the case of identical agent dynamics to the case of non-identical agent dynamics, and the other is we introduce pinning controllers to the selected agents.

The rest of this paper is organized as follows. A controlled continuous-time MAS model with communication time-delay is presented in Section II. The main results including pinning control and adaptive pinning control bounded consensus criterion are derived in Section III and V respectively. Section IV gives a numerical simulation example to verify the effectiveness of the proposed results, followed by conclusions in Section VI.

Wei-Song Zhong, Guo-Ping Liu, David Rees and Clive Thomas are with the Faculty of Advanced Technology, University of Glamorgan, Pontypridd, Mid Glamorgan, CF37 1DL U.K. e-mail: ({wzhong,gpliu,drees,ejcthomas}@glam.ac.uk).

Manuscript received April 19, 2005; revised January 11, 2007.

II. Problem description

Let $G = (\mathcal{V}, \mathcal{A})$ be a graph of order N consisting of a set of vertices $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and a set of edges $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$. An edge (v_j, v_i) in graph G means that agent v_i sends some information to agent v_j. The set of neighbors of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{A}\}$.

We consider a MAS consisting of N non-identical agents with communication delay:

$$\dot{x}_i = f_i(x_i) + c \sum_{j \in \mathcal{N}_i}^{N} a_{ij} \Gamma x_j(t - \tau), i = 1, 2, \cdots, N, \tag{1}$$

where $x_i = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ are the state variables of the agent v_i, $f_i(x_i) : R^n \to R^n$ are continuously differentiable mappings with Jacobian Df_i, representing the self-dynamics of the agent v_i, $c > 0$ denotes the coupling strength, $\Gamma = (\gamma_{ij}) \in R^{n \times n}$ is the inner coupling matrix, and where $\gamma_{ij} \neq 0$ means two connected agents are linked via their ith and jth state variables, respectively. The adjacency matrix $A = (a_{ij}) \in R^{N \times N}$ (which is symmetric and irreducible) represents the communication topology relation of the MAS, and is defined by $a_{ij} = a_{ji} = 1(v_j \in \mathcal{N}_i)$, $a_{ij} = 0(v_j \notin \mathcal{N}_i)$ and $a_{ii} = -\sum_{j \neq i} a_{ij}$. τ is a constant coupling delay which reflects the reality that the agent v_i can't obtain information from agent v_j instantaneously.

The average dynamic of all agents is defined by the vector field $\bar{f}(x(t)) = \frac{1}{N} \sum_{k=1}^{N} f_k(x(t))$ with Jacobian $D\bar{f}_i(x(t))$.

The average state trajectory is chosen as the desired moving trajectory

$$s(t) = \frac{1}{N} \sum_{k=1}^{N} x_k(t). \tag{2}$$

We now discuss the problem of global consensus for the system (1). The consensus problem formulation in the present paper is quite different from many others, where the consensus problem is solvable if the states of all agents satisfy $x_i(t) \to x_j(t)$, $\forall i,j=1,2,\cdots,N$ as $t\to\infty$. The consensus problem here will be depicted instead via certain boundedness of $x_i(t)-x_j(t)$, $\forall i,j=1,2,\cdots,N$ as $t\to\infty$. This better reflects reality as it is impossible for MAS (1) to achieve exact consensus. To address this case we will focus on making the states of all agents converge to a bounded set.

We denote $x(t)$, $s(t)$, $u(t)$, $e(t)$, $w(t)$ and $V(w(t),t)$ as x, s, u, e, w and V respectively.

III. Linear feedback pinning controller

To achieve the goal, we apply the feedback control strategy on a small fraction δ $(0<\delta\le 1)$ of the agents in system (1). Suppose that nodes $i_1,i_2,\cdots,i_l$ are selected to be under control, where $l=[\delta N]$ stands for the smaller but nearest integer to the real number δN. This controlled MAS can be described as

$$\begin{cases}\dot{x}_{i_k}=f_{i_k}(x_{i_k})+c\sum_{j=1}^N a_{i_kj}\Gamma x_j(t-\tau)+u_{i_k}, 1\le k\le l,\\ \dot{x}_{i_k}=f_{i_k}(x_{i_k})+c\sum_{j=1}^N a_{i_kj}\Gamma x_j(t-\tau), l+1\le k\le N.\end{cases} \tag{3}$$

The local linear negative feedback control law is chosen as follows:

$$\begin{cases}u_{i_k}=-d_{i_k}(x_{i_k}-s), & 1\le k\le l,\\ u_{i_k}=0, & l+1\le k\le N,\end{cases} \tag{4}$$

where the feedback gain $d_{i_k}>0$.

Combine (3) and (4) and rearrange the order of the nodes in the network. Let the first l nodes be controlled, and $e_i=x_i-s$, $i=1,2,\cdots,N$. It's obvious that $\frac{c}{N}\sum_{k=1}^N\sum_{j=1}^N a_{kj}\Gamma x_j(t-\tau)=0$ and $\sum_{i=1}^N e_i=0$. Then by applying the Newton-Leibniz formula, error systems can be written as

$$\begin{cases}\dot{e}_i=D\bar{f}(s)e_i+c\sum_{j=1}^N a_{ij}\Gamma e_j(t-\tau)\\ \quad+\int_0^1(Df_i(s+\tau e_i)-D\bar{f}(s))e_i d\tau\\ \quad-\frac{1}{N}\sum_{k=1}^N\int_0^1 Df_k(s+\tau e_k)e_k d\tau\\ \quad+f_i(s)-\bar{f}(s)-d_ie_i, & 1\le i\le l,\\ \dot{e}_i=D\bar{f}(s)e_i+c\sum_{j=1}^N a_{ij}\Gamma e_j(t-\tau)\\ \quad+\int_0^1(Df_i+\tau e_i)-D\bar{f}(s))e_i d\tau\\ \quad-\frac{1}{N}\sum_{k=1}^N\int_0^1 Df_k(s+\tau e_k)e_k d\tau\\ \quad+f_i(s)-\bar{f}(s), & l+1\le i\le N.\end{cases} \tag{5}$$

The following work will focus on simplifying the error systems (5) by means of a series of transformations using a procedure similar to [17].

Define the following matrix

$$D=diag(D_1,D_2,\cdots,D_N)\in R^{nN\times nN},$$

where $D_i=diag\{-d_i,-d_i,\cdots,-d_i\}\in R^{n\times n}$.

Let $e=(e_1^T,e_2^T,\cdots,e_N^T)^T$, then (5) becomes

$$\dot{e}=\bar{\Sigma}(t)e+cA\otimes\Gamma e(t-\tau)+I(t)e-\frac{1}{N}H(t)e+F(t), \tag{6}$$

where $I(t)=diag\{\int_0^1(Df_1(s+\tau e_1)-D\bar{f}(s))d\tau\cdots\int_0^1(Df_N(s+\tau e_N)-D\bar{f}(s))d\tau\}$, $\bar{\Sigma}(t)=I_N\otimes D\bar{f}(s)+D$, $H^T(t)=(H_1^T(t),\cdots,H_N^T(t))$, $H_i(t)=(\int_0^1 Df_1(s+\tau e_1)d\tau,\cdots,\int_0^1 Df_N(s+\tau e_N)d\tau)$, $F_i^T(t)=(f_1^T(s)-\bar{f}^T(s),\cdots,f_N^T(s)-\bar{f}^T(s))$.

Since A is symmetric and irreducible, according to [17], there exists a unitary matrix $\Phi=(\varphi_{ij})_{N\times N}=(\Phi_1,\Phi_2,\cdots,\Phi_N)$. This together with $w(t)=(\Phi^T\otimes I_n)e$ gives

$$\begin{aligned}\dot{w}&=(\Phi^T\otimes I_n)\bar{\Sigma}(t)(\Phi\otimes I_n)w\\&\quad+(\Phi^T\otimes I_n)(cA\otimes\Gamma)(\Phi\otimes I_n)w(t-\tau)\\&\quad+(\Phi^T\otimes I_n)I(t)(\Phi\otimes I_n)w\\&\quad-\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)w+(\Phi^T\otimes I_n)F(t).\end{aligned} \tag{7}$$

Note that $H(t)=\sqrt{N}\sum_{k=1}^N(\mathbf{0}\ \cdots\ \mathbf{0}\ \bar{\Phi}_k\ \mathbf{0}\ \cdots\ \mathbf{0})\otimes\int_0^1 Df_k(s+\tau e_k)d\tau$, where $\bar{\Phi}_k$ stands for the matrix with its k-th column equal to Φ_1 and the remaining elements are zero. Then we have $\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)=\frac{1}{\sqrt{N}}\sum_{k=1}^N(\mathbf{0}\cdots\mathbf{0}\ I_k\ \mathbf{0}\cdots\mathbf{0})\otimes\int_0^1 Df_k(s+\tau e_k)d\tau(\Phi\otimes I_n)$, where I_k stands for the matrix with its k-th column equals $(1\ 0\ \cdots\ 0)^T$ and the remaining of its elements are zero.

Thus, a simple calculation gives $\frac{1}{N}(\Phi^T\otimes I_n)H(t)(\Phi\otimes I_n)=\frac{1}{\sqrt{N}}\sum_{k=1}^N\begin{pmatrix}\Upsilon_k\\0\end{pmatrix}\otimes\int_0^1 Df_k(s(t)+\tau e_k(t))d\tau$, where $\Upsilon_k\in R^{1\times N}$ and $0\in R^{(N-1)\times N}$. Therefore, $\dot{w}=\bar{\Sigma}(t)w+c\Lambda\otimes\Gamma w(t-\tau)+(\Phi^T\otimes I_n)I(t)(\Phi\otimes I_n)w-\begin{pmatrix}*\\0\end{pmatrix}w+(\Phi^T\otimes I_n)F(t)$. Since $w_1\equiv 0$, we only need to consider $w_2,w_3,\cdots,w_N$. Rewriting in the component form we have

$$\begin{aligned}\dot{w}_i&=\Sigma_i(t)w_i+c\lambda_i\Gamma w_i(t-\tau)+(\Phi_i^T\otimes I_n)F(t)\\&\quad+(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w,\ i=2,3,\cdots,N,\end{aligned} \tag{8}$$

where $\Sigma_i=\bar{D}f(s)+D_i$.

So far, we have transferred the consensus problem of system (1) to the stability problem of the $N-1$ of $n-$dimensional systems.

Theorem 1 Suppose that $\|I(t)\|\le\gamma$ is satisfied. If there exist matrices $P_i(t)\in\mathcal{PC}^1_{n\times n}$, $Q_i>0$, $\Theta_i>0$, $\Pi_i>0$, X_i, Y_i and Z_i of appropriate dimensions such that

$$B=\begin{pmatrix}B_1 & B_2\\ B_2^T & B_3\end{pmatrix}<0,\ \begin{pmatrix}X_i & Y_i\\ Y_i^T & Z_i\end{pmatrix}\ge 0, \tag{9}$$

for $i=2,3,\cdots,N$, where $B_1=\dot{P}_i(t)+P_i(t)\Sigma_i(t)+\Sigma_i^T(t)P_i(t)+hX_i+Y_i^T+Y_i+Q_i+h\Sigma_i^T(t)Z_i\Sigma_i(t)$, $B_2=c\lambda_iP_i(t)\Gamma-Y_i+hc\lambda_i\Sigma_i^T(t)Z_i\Gamma$ and $B_3=\Pi_i^{-1}+\Theta_i^{-1}-Q_i+hc^2\lambda_i^2\Gamma^TZ_i\Gamma$, then the MAS (1) will achieve bounded consensus for the time-invariant delay $\tau\in[0,h]$ for some $h<\infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V=\sum_{i=2}^{N}\sum_{k=1}^{3}V_k, \tag{10}$$

where

$$\begin{aligned}V_1&=w_i^TP_i(t)w_i,\\ V_2&=\int_{-\tau}^{0}\int_{t+\beta}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha d\beta,\\ V_3&=\int_{t-\tau}^{t}w_i^T(\alpha)Q_iw_i(\alpha)d\alpha.\end{aligned}$$

The i-th $(i=2,3,\cdots,N)$ equation in system (8) can be written as

$$\begin{aligned}\dot{w}_i&=(\Sigma_i(t)+c\lambda_i\Gamma)w_i-c\lambda_i\Gamma\int_{t-\tau}^{t}\dot{w}_i(\alpha)d\alpha\\ &\quad+(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w+(\Phi_i^T\otimes I_n)F(t). \end{aligned}\tag{11}$$

Defining $a(.)$, $b(.)$ and M in [21] as $a(\alpha)=w_i(t)$, $b(\alpha)=\dot{w}_i(\alpha)$ and $M=c\lambda_iP_i(t)\Gamma$ for all $\alpha\in[t-\tau,t]$ then we have

$$\begin{aligned}\dot{V}_1\le\ & w_i^T[\dot{P}_i(t)+P_i(t)\Sigma_i(t)+\Sigma_i^T(t)P_i(t)\\ &+hX_i+Y_i^T+Y_i]w_i+\int_{t-\tau}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha\\ &+2w_i^T(c\lambda_iP_i(t)\Gamma-Y_i)w_i(t-\tau)\\ &+2w_i^TP_i(t)(\Phi_i^T\otimes I_n)I(t)(\Phi_i\otimes I_n)w\\ &+2w_i^TP_i(t)(\Phi_i^T\otimes I_n)F(t). \end{aligned}\tag{12}$$

Moreover, $\dot{V}_2$ can be enlarged as

$$\begin{aligned}\dot{V}_2\le\ & h[\Sigma_i(t)w_i+c\lambda_i\Gamma w_i(t-\tau)]^TZ_i[\Sigma_i(t)w_i\\ &+c\lambda_i\Gamma w_i(t-\tau)]+2h(\Sigma_i(t)w_i)^TZ_i(\Phi_i^T\otimes I_n)I(t)\\ &(\Phi\otimes I_n)w+2h(\Sigma_i(t)w_i)^TZ_i(\Phi_i^T\otimes I_n)F(t)\\ &+2h(c\lambda_i\Gamma w_i(t-\tau))^TZ_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w\\ &+2h(c\lambda_i\Gamma w_i(t-\tau))^TZ_i(\Phi_i^T\otimes I_n)F(t)\\ &+2h((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w)^TZ_i(\Phi_i^T\otimes I_n)F(t)\\ &+h((\Phi_i^T\otimes I_n)F(t))^TZ_i((\Phi_i^T\otimes I_n)F(t))\\ &+h((\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w)^TZ_i((\Phi_i^T\otimes I_n)I(t)\\ &(\Phi\otimes I_n)w)-\int_{t-\tau}^{t}\dot{w}_i^T(\alpha)Z_i\dot{w}_i(\alpha)d\alpha. \end{aligned}\tag{13}$$

and

$$\dot{V}_3=w_i^TQ_iw_i-w_i^T(t-\tau)Q_iw_i(t-\tau). \tag{14}$$

Applying the Young Inequality, then we have $2h(c\lambda_i\Gamma w_i(t-\tau))^TZ_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n)w\le w_i^T(t-\tau)\Pi_i^{-1}w_i(t-\tau)+h^2c^2\lambda_i^2w^T((\Phi\otimes I_n)^TI(t)(\Phi_i^T\otimes I_n)^TZ_i\Gamma\Pi_i\Gamma^TZ_i(\Phi_i^T\otimes I_n)I(t)(\Phi\otimes I_n))w(t)$, and $2h(c\lambda_i\Gamma w_i(t-\tau))^TZ_i(\Phi_i^T\otimes I_n)F(t)\le w_i^T(t-\tau)\Theta_i^{-1}w_i(t-\tau)+h^2c^2\lambda_i^2F^T(t)(\Phi_i^T\otimes I_n)^TZ_i\Gamma\Theta_i\Gamma^TZ_i(\Phi_i^T\otimes I_n)F(t)$. Applying these two inequalities and the conditions of the theorem results

$$\begin{aligned}\dot{V}\le\ & \sum_{i=2}^{N}\begin{pmatrix}w_i\\ w_i(t-\tau)\end{pmatrix}^TB\begin{pmatrix}w_i\\ w_i(t-\tau)\end{pmatrix}\\ &+2\mu(t)\beta+(\|w\|(2\gamma\beta+2h\gamma\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i)\\ &+2h\mu(t)\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i)+h\gamma^2\sum_{i=2}^{N}\lambda_{max}(Z_i)\\ &+h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i)\\ &+h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Theta_i)\lambda_i^2\lambda_{max}^2(Z_i))\|w\|\\ &+2h\gamma\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t))+h\mu^2(t)\sum_{i=2}^{N}\lambda_i^2\lambda_{max}(Z_i). \end{aligned}\tag{15}$$

Thus when

$$\|w\|\ge\frac{2\mu(t)\beta+2h\gamma\sum_{i=2}^{N}\lambda_{max}(Z_i)\mu(t)}{\varpi(t)},$$

we have

$$\dot{V}\le-\delta\|w\|^2+h\mu^2(t)\sum_{i=2}^{N}\lambda_{max}(Z_i)\lambda_i^2, \tag{16}$$

where $\varpi(t)=-(2\gamma\beta+2h\gamma\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i)+2h\mu(t)\|\Sigma_i(t)\|\sum_{i=2}^{N}\lambda_{max}(Z_i)+h\gamma^2\sum_{i=2}^{N}\lambda_{max}(Z_i)+h^2c^2\gamma^2\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Pi_i)\lambda_i^2\lambda_{max}^2(Z_i)+h^2c^2\mu^2(t)\lambda_{max}^{\frac{1}{2}}(\Gamma\Gamma^T)\sum_{i=2}^{N}\lambda_{max}(\Theta_i)\lambda_i^2\lambda_{max}^2(Z_i))-\delta$. Thus, according to [22] and Lyapunov stability theory, bounded consensus is ultimately achieved. This completes the proof.

IV. Adaptive pinning controller

In this section, we will derive globally consensus criteria via direct adaptive pinning control method. Without loss of generality, we still assume that the first l agents are selected as pinned agents with the adaptive controllers:

$$\begin{cases}u_i=-d_i(t)(x_i-s), & 1\le i\le l,\\ \dot{d}_i(t)=h_ie_i^TP_i(t)e_i, & \\ u_i=0, & l+1\le i\le N,\end{cases}\tag{17}$$

where constant $h_i>0$ and positive definite matrix $P_i(t)\in R^{n\times n}$. Applying Newton-Leibniz formula, then the error MAS can be rewritten as

$$\begin{cases}\dot{e}_i=D\bar{f}(s)e_i+c\sum_{j=1}^{N}a_{ij}\Gamma e_j(t-\tau)\\ \quad+\int_0^1(Df_i(s+\tau e_i)-D\bar{f}(s))e_id\tau\\ \quad-\frac{1}{N}\sum_{k=1}^{N}\int_0^1Df_k(s+\tau e_k)e_kd\tau\\ \quad+f_i(s)-\bar{f}(s)-d_i(t)e_i, \qquad 1\le i\le l,\\ \dot{d}_i(t)=h_ie_i^TP_i(t)e_i,\\ \dot{e}_i=D\bar{f}(s)e_i+c\sum_{j=1}^{N}a_{ij}\Gamma e_j(t-\tau)\\ \quad+\int_0^1(Df_i(s+\tau e_i)-D\bar{f}(s))e_id\tau\\ \quad-\frac{1}{N}\sum_{k=1}^{N}\int_0^1Df_k(s+\tau e_k)e_kd\tau\\ \quad+f_i(s)-\bar{f}(s), \qquad l+1\le i\le N.\end{cases}\tag{18}$$

Repeating a similar procedure to the previous subsection, the controlled consensus problem of system (1) is equivalent to the stability problem of the following $N-1$ of n-dimensional systems.

$$\begin{cases} \dot{w}_i = D\bar{f}(s(t))w_i - d_i(t)w_i + c\lambda_i\Gamma w_i(t-\tau) \\ \qquad +(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \qquad +(\Phi_i^T \otimes I_n)F(t), \qquad 2 \le i \le l, \\ \dot{d}_i(t) = h_i w_i^T P_i(t) w_i, \\ \dot{w}_i = D\bar{f}(s)w_i + c\lambda_i\Gamma w_i(t-\tau) \\ \qquad +(\Phi_i^T \otimes I_n)I(t)(\Phi \otimes I_n)w \\ \qquad +(\Phi_i^T \otimes I_n)F(t), \qquad l+1 \le i \le N, \end{cases} \tag{19}$$

where w_i, w, Φ, Φ_i, $I(t)$ and $F(t)$ are the same as the previous subsection.

Theorem 2 Suppose that $\|I(t)\| \le \gamma$ is satisfied. If there exist matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$, $Q_i > 0$, $\Theta_i > 0$, $\Pi_i > 0$, X_i, Y_i and Z_i of appropriate dimensions and constant $d > 0$ such that

$$B = \begin{pmatrix} B_1 & B_2 \\ B_2^T & B_3 \end{pmatrix} < 0, \quad \begin{pmatrix} X_i & Y_i \\ Y_i^T & Z_i \end{pmatrix} \ge 0, \tag{20}$$

for $i = 2, 3, \cdots, N$, where $B_1 = \dot{P}_i(t) + P_i(t)(Df(s)) + (Df(s))^T P_i(t) - 2dP_i(t) + hX_i + Y_i^T + Y_i + Q_i + h\Sigma_i^T(t)Z_i\Sigma_i(t)$, $B_2 = c\lambda_i P_i(t)\Gamma - Y_i + hc\lambda_i\Sigma_i^T(t)Z_i\Gamma$ and $B_3 = \Pi_i^{-1} + \Theta_i^{-1} - Q_i + hc^2\lambda_i^2\Gamma^T Z_i\Gamma$, then the system (1) will achieve bounded consensus for the time-invariant delay $\tau \in [0, h]$ for some $h < \infty$.

Proof. Construct the following Lyapunov-Krasovskii functional as

$$V = \sum_{i=2}^{N}\sum_{k=1}^{3} V_k + \sum_{i=2}^{l} \frac{(d_i(t)-d)^2}{h_i}, \tag{21}$$

where

$$V_1 = w_i^T P_i(t) w_i,$$
$$V_2 = \int_{-\tau}^{0}\int_{t+\beta}^{t} \dot{w}_i^T(\alpha) Z_i \dot{w}_i(\alpha) d\alpha d\beta,$$
$$V_3 = \int_{t-\tau}^{t} w_i^T(\alpha) Q_i w_i(\alpha) d\alpha.$$

The remaining part of the proof is similar to that of Theorem 1 and is therefore omitted here. This completes the proof.

V. Example

To demonstrate the theoretical results obtained above, we construct a MAS consisting of 11 agents described as follows

$$\dot{x}_i(t) = f_i(x_i(t)) + c\sum_{j\in\mathcal{N}_i}^{N} a_{ij}\Gamma x_j(t-\tau), \tag{22}$$

where $f_i(x_i(t)) = B_i x_i(t) + g(x_i(t))$, $B_i(i = 1, 2, \cdots, 6)$ and $B_i(i = 7, 8, \cdots, 11)$ are chosen as follows:

$$\begin{pmatrix} -10+0.1\times(i-1) & 10-0.1\times(i-1) & 0 \\ 1 & -1 & 1 \\ 0 & -15-0.1\times(i-1) & 0 \end{pmatrix},$$
$$\begin{pmatrix} -10-0.1\times(i-6) & 10+0.1\times(i-6) & 0 \\ 1 & -1 & 1 \\ 0 & -15+0.1\times(i-6) & 0 \end{pmatrix},$$

and

$$g(x_i(t)) = (-9.5sin(\frac{\pi x_{i1}(t)}{3.2} + \pi)\ 0\ 0)^T, \quad i = 1, 2, \cdots, 11.$$

The communication coupling matrix $C = (C_1^T C_2^T \cdots C_{11}^T)$, $C_1 = (-8\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1)$, $C_2 = (1\ -8\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1)$, $C_3 = (1\ 1\ -6\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1)$, $C_4 = (0\ 1\ 1\ -5\ 0\ 1\ 1\ 0\ 0\ 1\ 0)$, $C_5 = (1\ 1\ 0\ 0\ -6\ 0\ 1\ 1\ 1\ 1\ 0)$, $C_6 = (1\ 0\ 0\ 1\ 0\ -5\ 1\ 0\ 1\ 1\ 0)$, $C_7 = (1\ 1\ 0\ 1\ 1\ 1\ -7\ 1\ 0\ 1\ 0)$, $C_8 = (0\ 0\ 1\ 0\ 1\ 0\ 1\ -5\ 0\ 1\ 1)$, $C_9 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ -6\ 1\ 1)$, $C_{10} = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ -10\ 1)$, $C_{11} = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ -6)$. $\Gamma = diag\{2, 2, 2\}$, respectively, where the matrix A is produced by means of the Scale-Free network program.

Design the following controllers

$$\begin{cases} u_{i_k} = -d_{i_k}(x_{i_k}(t) - s(t)), & i_k = 1, 2 \text{ and } 10, \\ u_{i_k} = 0, & \text{else}, \end{cases}$$

with $d_1 = 0.5$, $d_2 = 0.5$, $d_{10} = 0.5$ and

$$\begin{cases} u_{i_k} = -d_{i_k}(t)(x_{i_k}(t) - s(t)), & i_k = 1, 2 \text{ and } 10, \\ \dot{d}_{i_k}(t) = h_{i_k} e_{i_k}^T P_{i_k}(t) e_{i_k}, \\ u_{i_k} = 0, & \text{else}, \end{cases}$$

with $h_1 = 0.1$, $h_2 = 0.2$, $h_{10} = 0.3$, $s(t)$ can then be evaluated by simulation.

Given the initial values of 11 agents as $(10\ 5\ -10)^T$, $(12\ 6\ -12)^T$, $(14\ 7\ -14)^T$, $(16\ 8\ -16)^T$, $(18\ 9\ -18)^T$, $(20\ 10\ -20)^T$, $(-18\ 11\ 18)^T$, $(-16\ 12\ 16)^T$, $(-14\ 13\ 14)^T$, $(-12\ 14\ 12)^T$, $(-10\ 15\ 10)^T$ respectively and $P_{i_k}(t) = I_3$. We may verify the conditions of Theorem 1 and Theorem 2 readily. This demonstrates the bounded consensus of the MAS is achieved for any time delay $0 < \tau \le 0.061$. Simulation results are depicted in Fig.1 to Fig.4 for $\tau = 0.061$ and $c = 1$.

VI. Conclusion

In this paper, we've investigated the controlled consensus problems of MAS with different agent dynamics. The derived criteria are verified via theoretical analysis and numerical simulation. The consensus for the MAS is achieved based on pinning control and adaptive pinning control methods. It should be noted that the conditions are still restrictive and all the delays are the same. Further investigations will focus on relaxing these limitations.

VII. Acknowledgment

The authors would like to thank the anonymous reviewers for their valuable suggestions and comments.

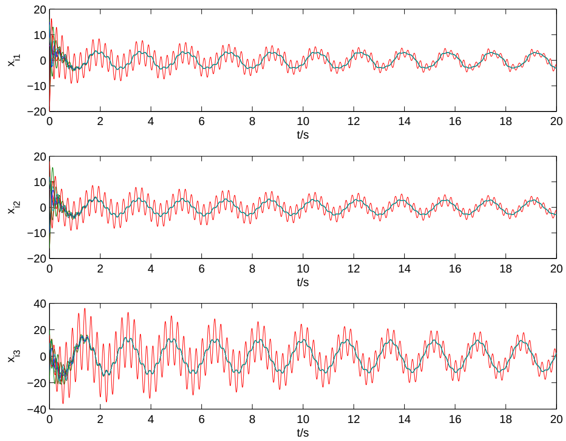

Fig.1. All agent dynamics under pinning control.

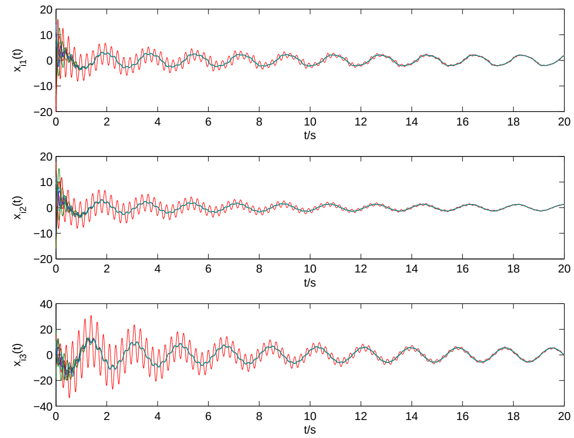

Fig.2. All agent dynamics under adaptive pinning control.

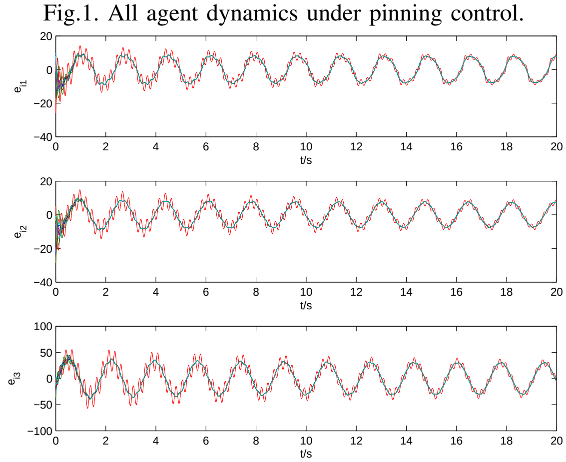

Fig.3. All agent error dynamics under pinning control.

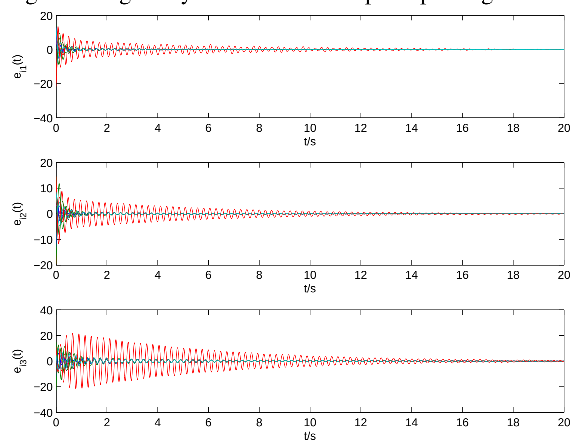

Fig.4. All agent error dynamics under adaptive pinning control.

REFERENCES

[1] Sandro Zampieri. Trends in networked control systems[C], *Proceedings of 17th World Congress of IFAC*, Seoul, Korea, July 6-11, 2008, pp. 2886 – 2894.

[2] J.P. Desai, J.P. Ostrowski and V. Kumar. Modeling and control of formation of nonholonomic mobile robust[J]. *IEEE Int. Trans. on Robotics and Automation*, 2001, 17(6): 905 – 908.

[3] M. Porfiri, D.G. Roberson and D.J. Stilwell. Tracking and formation control of multiple autonomous agents: a two-level consensus approach[J]. *Automatica*, 2007, 43: 1318 – 1328.

[4] J. Cortés. Global formation-shape stabilization of relative sensing networks[C]. *2009 American Control Systems*, Hyatt Regency Riverfront, St. Louis, USA, June 10-12, 2009, pp. 1460 – 1465.

[5] Y.G. Hong, J.P. Hu and L.X. Gao. Tracking control for multi-agent consensus with an active leader and variable topology[J]. *Automatica*, 2006, 42: 1177 – 1182.

[6] F. Xiao and L. Wang. Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays[J]. *IEEE Transactions on Automatic Control*, 2008, 53(8): 1804 – 1816.

[7] E.S. Kazerooni and K. Khorasani. Optimal consensus algorithms for cooperative team of agents subject to partial information[J]. *Automatica*, 2008, 44: 2766 – 2777.

[8] Z.K. Li, Z.S. Duan, G.R. Chen and L. Huang. Consensus of multiagent systems and synchronization of complex networks: a unified viewpoint[J]. *IEEE Transactions on Circuits and Systems-I*, 2010, 57(1): 213 – 224.

[9] R. Olfati-Saber and R.M. Murray. Consensus problmes in networks with switching topology and time-delays[J]. *IEEE Transactions on Automatic Control*, 2004, 49(9): 1520 – 1533.

[10] M. Arack. Passivity as a design tool for group coordination[J]. *IEEE Transactions on Automatic Control*, 2007, 52(8): 1380 – 1390.

[11] F. Chen, Z.Q. Chen, L.Y. Xiang, Z.X. Liu and Z.Z. Yuan. Reaching a consensus via pinning control[J]. *Automatica*, 2009, 45: 1215 – 1220.

[12] Y.P. Tian and C.L Liu. Robust consensus of multi-agent systems with diverse input delays and asymmetric interconnection perturbations[J]. *Automatica*, 2009, 45: 1347 – 1353.

[13] P. Lin, Y.M. Jia and L. Li. Distributed robust H_∞ Consensus control in directed networks of agents with time-delay[J]. *Systems and Control Letters*, 2008, 57(8): 643 – 653.

[14] P.A. Bliman and G.F. Trecate. Average consensus problems in networks of agents with delayed communications[J]. *Automatica*, 2008, 44: 1985 – 1995.

[15] J. Cortes. Distributed algorithms for reaching consensus on general functions[J]. *Automatica*, 2008, 44(3): 726 – 737.

[16] Z.S Duan and G.R. Chen. Global Robust Stability and Synchronization of Networks With Lorenz-Type Nodes [J]. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, 2009, 56(8): 679 – 683.

[17] D. J.Hill and J. Zhao. Global synchronization of complex dynamical networks with non-identical nodes[C], *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, Dec. 9-11, 2008, 817 – 822.

[18] J.C. Zhao, J.A. Lu and Q.J. Zhang. Pinning a complex dynamical network to a homogenous trajectory[J]. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, 2009, 56(6): 514 – 518.

[19] W. Wu, W.J. Zhou and T.P. Chen. Cluster Synchronization of Linearly Coupled Complex Networks Under Pinning Control[J]. *IEEE Transactions on Circuits and Systems-I*, 2009, 56(4): 829 – 839.

[20] Y.Y. Wu, W. Wei, G.Y. Li and J. Xiang. Pinning Control of Uncertain Complex Networks to a Homogeneous Orbit [J]. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, 2009, 56(3): 235 – 239.

[21] Y.M. Moon, P. Park, W.H. Kwon, Y.S. Lee. Delay-dependent robust stabilization of uncertain state-delayed systems[J]. *Int J. Control*, 2001, 74: 1447 – 1455.

[22] C.C. Hua, X.P. Guan and P. Shi. Decentralized robust model reference adaptive cotnrol for interconnected time-delay systems[J]. *J.Comput.Appl.Math.*, 2006, 193: 383 – 396.